Journal of Statistical Physics, Vol. 22, No. 2, 1980

The FKG Inequality for the Yukawa$_2$ Quantum Field Theory

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Received May 25, 1979; revised July 24, 1979

We establish the FKG correlation inequality for the Euclidean scalar Yukawa$_2$ quantum field model and, when the Fermi mass is zero, for pseudoscalar Yukawa$_2$. To do so we approximate the quantum field model by a lattice spin system and show that the FKG inequality for this system follows from a positivity condition on the fundamental solution of the Euclidean Dirac equation with external field. We prove this positivity condition by applying the Vekua–Bers theory of generalized analytic functions.

KEY WORDS: FKG correlation inequality; Yukawa$_2$ model; quantum field theory; generalized analytic functions; Euclidean Dirac equation.

1. INTRODUCTION

Correlation inequalities have played a significant role in both statistical mechanics and quantum field theory. Consider in particular the problem of taking the thermodynamic limit of the correlation functions, or, in the language of Euclidean field theory, the infinite-volume limit of the Schwinger functions. One expects expansion techniques to succeed for weakly coupled models, where by "weakly coupled" we mean high temperature or small coupling constant, or low activity or large external field. As evidence for this expectation see Ruelle,[1] Glimm *et al.*,[2] or Spencer.[3] On the other hand, monotonicity arguments involving correlation inequalities are applicable regardless of the strength of the coupling. For example, for the $P(\phi)_2$ quantum

Research partially supported by the National Research Council of Canada.

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field model, the existence of the infinite-volume limit follows from the use of Griffiths inequalities[4–6] provided the polynomial P is *even*. Recently, Fröhlich and Simon[7] have taken the infinite-volume limit for *arbitrary* (semibounded) P by using the FKG inequality.[8,5] Actually, their construction also involves expansion techniques, since they begin with Spencer's large-external-field result,[3] and they then "turn off" the external field by means of the FKG inequality.

From the point of view of classical statistical mechanics the main result of this paper is that the FKG inequality holds for a certain two-dimensional lattice spin system with a rather complicated interaction. This theorem is of interest to constructive quantum field theory because, and this is one of our major results, the continuum limit of this lattice model is the Euclidean Yukawa$_2$ (Y_2) model in the Matthews–Salam–Seiler formulation. The FKG inequality carries over to this limit and so we may apply the method of Simon and Fröhlich for constructing a (partial) infinite-volume theory for the strongly coupled Y_2 model.[9] The weakly coupled Y_2 model has already been constructed since the Glimm–Jaffe–Spencer cluster expansion has been carried out for small coupling constant[10,11] and for large external field.[12]

The Yukawa$_2$ quantum field model describes the interaction of scalar or pseudoscalar bosons with spin-$\frac{1}{2}$ fermions in two space–time dimensions. There have been very few correlation inequalities proven for this model. For pseudoscalar Y_2 the first Griffiths inequality has been established by MacDermot[13] (see also Nicolai[14] and Rosen[15]), but we are dubious that the more useful second Griffiths inequality holds (at least for scalar Y_2).

It is important to understand why the presence of fermions in the Y_2 model poses difficulties in the search for correlation inequalities. In a purely. bosonic model such as $P(\phi)_2$, the theory with interaction in a finite $\Lambda \subset \mathbb{R}^2$ is specified by a probability measure on $S'(\mathbb{R}^2)$ of the form

$$d\nu(\phi) = \text{const}\ \rho(\phi)\, d\mu(\phi) \tag{1.1}$$

where $d\mu(\phi)$ is the free boson measure, i.e., the Gaussian measure in the field $\phi(x)$ with mean zero and covariance $C = (-\Delta + m_b{}^2)^{-1}$, where $m_b > 0$ is the boson mass, and where $\rho = e^{-U_\Lambda}$, with U_Λ a local function of ϕ, e.g., $U_\Lambda = \int_\Lambda :P(\phi(x)): dx$. By the lattice approximation of Guerra *et al.*[5] it is possible to approximate such a field theory model by a statistical mechanical ferromagnetic spin model on a lattice, as we now describe: Let L denote the rectangular lattice of points in Λ with spacing $\delta > 0$, i.e., $L = \{j\delta \mid j \in Z^2, j\delta \in \Lambda\}$. At each site $j\delta \in L$ one introduces a spin variable $q_j \in \mathbb{R}$ (roughly speaking, the average of ϕ in the square Δ_j with center $j\delta$ and sides of length δ; see Section 4). One defines the lattice cutoff field in Λ by

$$\phi_\delta(x) = \sum_j \chi_{\Delta_j}(x) q_j \tag{1.2}$$

and the lattice approximation to ρ by

$$\rho_\delta = \exp\left[-\int_\Lambda :P(\phi_\delta(x)):\, dx\right]$$

As far as expectations of functions of the q_j are concerned, the corresponding approximate measure const $\rho_\delta\, d\mu$ reduces to a probability measure

$$d\nu_\delta(q) = \text{const } \rho_\delta(q) \exp(-\tfrac{1}{2} q\cdot Aq) d_q^n \tag{1.3}$$

on $\mathbb{R}^n$, with n the number of sites in L, where (i) $\rho_\delta(q)$ is *local* in the sense that $\rho_\delta(q) = \prod_j \rho_j(q_j)$, so that ρ_δ does not couple spins at different sites; (ii) the Gaussian part of (1.3) is *ferromagnetic* in the sense that $A_{ij} \leqslant 0$ if $i \neq j$.

One then establishes[5] correlation inequalities for the statistical mechanical model with measure (1.3). These inequalities survive in the limit $\delta \to 0$ because of the basic convergence result

$$\rho_\delta \to \rho \qquad \text{in} \quad L^p(d\mu) \tag{1.4}$$

for any $p < \infty$.

Unfortunately, when fermions are involved there simply is not a suitable measure like (1.1) on which to base a proof of correlation inequalities (in spite of the efforts of many people). However, it was an important discovery of Seiler[16] that, if the fermion fields are integrated out by a formal procedure of Matthews and Salam, then one is left with a well-defined measure in the boson field, of the form (1.1). This raises the possibility of proving correlation inequalities involving functions of the boson field alone. One pays a price, though, for the Matthews–Salam–Seiler formalism, namely, that the density ρ is a complicated *nonlocal* function of the field ϕ [see (1.9) below]. Thus the spin model that results from a lattice approximation has a rather peculiar nonlocal measure. Still, we are able to establish the FKG inequality for such a complicated system because the FKG condition on $\mathbb{R}^n$ follows from a very simple sufficient condition [(1.5) below]. We should mention that there is an alternate approach to correlation inequalities for models involving fermions, due to Fröhlich and Park[17]: instead of "integrating out" the fermions, they "bosonize" the fermions, i.e., they express bilinear quantities in the fermion field in terms of an auxiliary boson field. In this way they obtain first and second Griffiths inequalities for a regularized version of the Y_2 model. Modulo the technical problem of removing the regularization, their inequalities lead to results on the existence of the infinite-volume limit that are very similar to ours.

The condition (1.5) in the following version of the FKG inequality on $\mathbb{R}^n$ has been used by Sax[18] and Avron *et al.*[19] and demystifies (for us) the proof of the FKG inequality:

Theorem 1.1 (FKG Inequality). Let $d\nu = e^w\, d^n q$ be a probability measure on $\mathbb{R}^n$ with $w \in C^2(\mathbb{R}^n)$. Suppose that

$$\partial^2 w/\partial q_i\, \partial q_j \geqslant 0, \qquad i \neq j \tag{1.5}$$

Then

$$\int fg\, d\nu - \int f\, d\nu \int g\, d\nu \geqslant 0 \tag{1.6}$$

for all increasing functions f and g on $\mathbb{R}^n$ (for which f, g, and fg are $d\nu$-integrable).

Recall that an increasing function f on $\mathbb{R}^n$ is one for which $f(x) \leqslant f(y)$ if $x_i \leqslant y_i$, $i = 1,..., n$.

Before proving this theorem we should comment further on its background. This result may not look quite like the familiar FKG theorem,[20–22] where the hypotheses are stated in this way: $d\nu = \rho\, d^n q$ is assumed to be a probability measure on $\mathbb{R}^n$ whose density ρ satisfies

$$\rho(p \vee q)\rho(p \wedge q) \geqslant \rho(p)\rho(q) \tag{1.7}$$

where $(p \vee q)_i = \max(p_i, q_i)$ and $(p \wedge q)_i = \min(p_i, q_i)$. But the hypotheses (1.5) and (1.7) are essentially equivalent (see Appendix A). We should also point out that the proof of our theorem remains valid if $d^n q$ is replaced by an arbitrary product measure on $\mathbb{R}^n$. It follows that one can easily recover from Theorem 1.1 the FKG inequality on a finite distributive lattice whose measure satisfies (1.7).[20] Finally, we mention that Glimm and Jaffe[23] have given a similar proof of a special case of Theorem 1.1.

Proof. The proof is by induction on n. The case $n = 1$ is trivial because the left side of (1.6) can be written as

$$\frac{1}{2}\int\int [f(p) - f(q)][g(p) - g(q)]\, d\nu(p)\, d\nu(q)$$

which is nonnegative because the integrand is. Assume now that the theorem is true on $\mathbb{R}^{n-1}$ and consider the inequality on $\mathbb{R}^n$. We write $q = (q', q_n)$, where $q' \in \mathbb{R}^{n-1}$ and, by Fubini's theorem, we condition on q_n; i.e., we write

$$\int fg\, d\nu = \int dq_n\, \rho(q_n) \int f(q', q_n) g(q', q_n)\, d\nu_{q_n}(q')$$

where the probability measure $d\nu_{q_n} = \rho(q_n)^{-1} \exp[w(q', q_n)]\, dq'$ and $\rho(q_n) = \int \exp[w(q', q_n)]\, dq'$. The inductive hypothesis clearly applies to the q' integral so that

$$\int fg\, d\nu \geqslant \int dq_n\, \rho(q_n) F(q_n) G(q_n) \tag{1.8}$$

where $F(q_n) = \int f(q', q_n)\, d\nu_{q_n}$ and $G(q_n) = \int g(q', q_n)\, d\nu_{q_n}$. Since, as we show next, F and G are increasing, we deduce from (1.8) and the $n = 1$ result that (1.7) holds:

$$\int fg\, d\nu \geqslant \int dq_n\, \rho(q_n)F(q_n) \int dq_n\, \rho(q_n)G(q_n) = \int f\, d\nu \int g\, d\nu$$

Now, because f is increasing, to show that $F(q_n)$ is increasing it suffices to show that $\tilde{F}(q_n) \equiv \int f(q', q_0)\, d\nu_{q_n}$ is increasing in q_n for fixed q_0. We compute that

$$\frac{d\tilde{F}}{dq_n} = \int \frac{\partial w}{\partial q_n}(q', q_n) f(q', q_0)\, d\nu_{q_n} - \int \frac{\partial w}{\partial q_n}(q', q_n)\, d\nu_{q_n} \int f(q', q_0)\, d\nu_{q_n}$$

(Strictly speaking, the factor $\partial w/\partial q_n$ may result in divergent integrals, but we can always first regularize by truncating f and g and the region of integration and then take limits.) Obviously (1.5) implies that $(\partial w/\partial q_n)(q', q_n)$ is an increasing function of q' and so by the inductive hypothesis $d\tilde{F}/dq_n \geqslant 0$. Hence F and G are increasing. ■

We now define the Y_2 model in the Matthews–Salam–Seiler formulation.[16] The measure $d\nu(\phi)$ on $S'(\mathbb{R}^2)$ for the interacting theory in a finite volume $\Lambda \subset \mathbb{R}^2$ has the form (1.1), where

$$\rho = \rho(K) = c\, \det{}_3(1 - \lambda K) \exp(-\lambda^2 B) \tag{1.9}$$

The explanation of (1.9) is as follows. c is a positive constant chosen so that $\int \rho\, d\mu = 1$; K is an operator on the Hilbert space

$$\begin{aligned} \mathscr{H} = \mathscr{H}_{m_f} &= \{(f_0, f_1) | (p^2 + m_f^2)^{1/4} \hat{f}_i(p) \in L^2(\mathbb{R}^2)\} \\ &= L^2((p^2 + m_f^2)^{1/2}\, d^2p) \otimes \mathbb{C}^2 \end{aligned} \tag{1.10}$$

where $m_f \geqslant 0$ is the Fermi mass (where no confusion may arise, we shall drop the subscripts b and f on m). The operator K is defined, at least formally, by its integral kernel

$$K(x, y) = S(x, y)\phi(y)\chi_\Lambda(y) \tag{1.11}$$

where ϕ is the boson field, χ_Λ is the characteristic function of Λ, and S is the free Fermi two-point function:

$$S(x, y) = \frac{1}{(2\pi)^2} \int d^2p\, e^{ip(x-y)} \frac{\not{p} + m}{p^2 + m^2} \Gamma \tag{1.12}$$

with

$$\not p = i\beta_0 p_0 + i\beta_1 p_1, \qquad \beta_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \beta_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.13}$$

$$\Gamma = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{scalar } Y_2 \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \text{pseudoscalar } Y_2 \end{cases} \tag{1.14}$$

We shall set the coupling constant $\lambda = 1$ throughout this paper. The subtracted determinant is defined by[24]

$$\det_3(1 - K) = \exp[\mathrm{Tr}\ln(1 - K) + K + K^2/2] \tag{1.15}$$

and B is defined when $m_f > 0$ by

$$B = \tfrac{1}{2}{:}\mathrm{Tr}(K^2 + K^\dagger K){:} \tag{1.16}$$

where $K^\dagger$ is the adjoint of K on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and : : denotes Wick ordering with respect to $d\mu$. When $m_f = 0$ the expression (1.16) suffers an infrared divergence and we must modify the definition (see Section 7).

The above formalism has been justified by Seiler,[16] who showed in particular that, for $m_f > 0$, $\rho \in L^p(d\mu)$ for $p < \infty$. In Section 2 we formulate and prove two "abstract" theorems (Theorems 2.1 and 2.2) which isolate the estimates needed to recover Seiler's L^p result. These theorems contain nothing new for the expert on the Y_2 model, but they enable us to carry out the approximations required in this paper. As a further application of Theorem 2.2, we analyze the Y_2 model with $m_f = 0$ in Section 7.

We now ask whether the Y_2 measure $d\nu$ in (1.1) satisfies (1.5). Of course $S'(\mathbb{R}^2)$ is not $\mathbb{R}^n$, but one should bear in mind that the theorem is to be applied to the approximating lattice spin system. We should obtain the correct intuition if we proceed *formally* in the best traditions of quantum field theory with functional derivatives $\delta/\delta\phi(x)$ replacing partial derivatives $\partial/\partial q_i$. By (1.1) and (1.9)

$$\begin{aligned} d\nu &= \text{const}\exp[-\tfrac{1}{2}(\phi, (-\Delta + m_b{}^2)\phi) + \mathrm{Tr}\, K - \tfrac{1}{2}{:}\mathrm{Tr}\, K^\dagger K{:} \\ &\qquad + \mathrm{Tr}\ln(1 - K)] \prod_{x\in\mathbb{R}^2} d\phi(x) \\ &\equiv \text{const}\,[\exp W(\phi)] \prod d\phi(x) \end{aligned}$$

The inequality we desire is

$$\delta^2 W/\delta\phi(x)\,\delta\phi(y) \geqslant 0, \qquad x \neq y \tag{1.17}$$

Now, the mixed second partial of the first term in W is just the corresponding off-diagonal element of the "matrix" $\Delta - m^2$, which is positive for x infinitesimally close to y and zero otherwise. This heuristic statement is made rigorous by the lattice approximation of Ref. 5. The next two terms in W do not contribute to (1.17), since $\operatorname{Tr} K$ is linear in ϕ and $:\operatorname{Tr} K^{\dagger}K: = \delta m^2 \int :\phi(x)^2: dx$ is a local function of ϕ. Therefore (1.17) reduces to showing that

$$\frac{\delta^2}{\delta\phi(x)\,\delta\phi(y)} \operatorname{Tr} \ln(1 - K) \geqslant 0, \qquad x \neq y$$

For $x, y \in \Lambda$ we calculate using the definition (1.11) that

$$\frac{\delta}{\delta\phi(y)} \operatorname{Tr} \ln(1 - K) = -\operatorname{Tr}(1 - K)^{-1} \frac{\delta K}{\delta\phi(y)} = -\operatorname{Tr}(1 - K)^{-1} S\delta_y$$

and

$$\frac{\delta^2}{\delta\phi(x)\,\delta\phi(y)} \operatorname{Tr} \ln(1 - K) = -\operatorname{Tr}(1 - K)^{-1} S\delta_x (1 - K)^{-1} S\delta_y$$

where $\delta_y(z) = \delta(z - y)$. In terms of

$$S' \equiv (1 - K)^{-1} S \tag{1.18}$$

we conclude that

$$\frac{\delta^2}{\delta\phi(x)\,\delta\phi(y)} \operatorname{Tr} \ln(1 - K) = -\operatorname{tr} S'(y, x) S'(x, y)$$

where tr denotes the trace on 2×2 matrices, the integration over the continuous variables having been eliminated by the δ functions. Thus (1.17) reduces to

$$\operatorname{tr} S'(y, x) S'(x, y) \leqslant 0, \qquad x \neq y \tag{1.19}$$

From the definition (1.18), we see that S' is the fundamental solution (vanishing at infinity) of the external field problem

$$[\Gamma^{-1}(-\beta \cdot \partial_x + m_f) - \phi(x)\chi_\Lambda(x)] S'(x, y) = \delta(x - y) \tag{1.20}$$

As so often happens, the field-theoretic problem has (formally) reduced to the problem of establishing an estimate (1.19) for a classical Green's function. To analyze S' we recast the problem in complex notation: we set $z = x_0 + ix_1$ so that

$$\partial_z = \frac{1}{2}\left\{\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1}\right\}, \qquad \partial_{\bar{z}} = \frac{1}{2}\left\{\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}\right\}$$

and we identify real two-component spinors with complex functions

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv u_1 + iu_2$. Then the homogeneous equation corresponding to (1.20) becomes

$$\partial_{\bar{z}} u + B\bar{u} = 0 \tag{1.21}$$

where

$$B(z) = \begin{cases} \frac{1}{2}i(\phi\chi_\Lambda - m_f) & \text{scalar} \\ \frac{1}{2}(\phi\chi_\Lambda - im_f) & \text{pseudoscalar} \end{cases} \tag{1.22}$$

We are thus dealing with a perturbed Cauchy–Riemann equation. Such equations have been extensively studied by Vekua[(25)] and Bers,[(26)] the solutions u of (1.21) being called *generalized analytic functions.* We review the relevant results of this theory in Section 3. The central result of the theory is that every generalized analytic function can be represented as a product of an analytic function with a bounded, continuous, nonvanishing function (see Lemma 3.4). This type of representation will enable us to prove (1.19) for $m_f \geqslant 0$ in the scalar case, but only for $m_f = 0$ in the pseudoscalar case. It should be pointed out that Vekua[(25)] places certain regularity conditions on B both locally and at infinity. To make contact with Vekua's theory we must accordingly regularize the B in (1.22).

This completes our heuristic description of the ideas involved in proving the main theorem of this paper:

Theorem 1.2. Consider the scalar Y_2 model with $m_f \geqslant 0$ or the pseudoscalar model with $m_f = 0$ with interaction in the rectangle Λ. Let $d\nu$ be the normalized measure defined by (1.1) and (1.9). Let $\mathscr{I}$ be the class of increasing functions of the boson field,

$$\begin{aligned} \mathscr{I} = \{F(\phi(h_1),\dots,\phi(h_r)) | r \geqslant 0;& \\ F\colon \mathbb{R}^r \to \mathbb{R} \text{ continuous and increasing};& \\ h_j \in L^2(\Lambda),\ h_j \geqslant 0\}& \end{aligned}$$

Then for all $F, G \in \mathscr{I}$

$$\int FG\, d\nu \geqslant \int F\, d\nu \int G\, d\nu \tag{1.23}$$

provided F, G, and FG are $d\nu$-integrable.

It takes an embarrassing amount of technical work for us to convert the above heuristics to a rigorous proof. Perhaps an outline at this point would help. Our first step is to place the lattice cutoff (1.2) on the boson field. We define the lattice cutoff density to be [compare with (1.9)]

$$\rho_\delta = c_\delta \rho(K_\delta) = c_\delta \det{}_3(1 - K_\delta)\exp(-B_\delta) \tag{1.24}$$

where

$$K_\delta(x, y) = S(x, y)\phi_\delta(y)\chi_\Lambda(y) \tag{1.25}$$

$B_\delta = \frac{1}{2}{:}\mathrm{Tr}(K_\delta^2 + K_\delta^\dagger K_\delta){:}$, and the constant c_δ is chosen so that $\int \rho_\delta \, d\mu = 1$. Section 4 is devoted to the proof that

$$\rho_\delta \to \rho \quad \text{in} \quad L^p(d\mu) \tag{1.26}$$

for any $p < \infty$, at least for a subsequence $\delta_n \to 0$. As far as expectations of functions of the q_j are concerned, the measure $\rho_\delta \, d\mu$ reduces to a measure of the form (1.3) on $\mathbb{R}^n$ except that the non-Gaussian factor ρ_δ is nonlocal. Thus, as in the case of $P(\phi)_2$, we have cut down the continuum of random variables $\{\phi(x)\}_{x \in \Lambda}$ to a finite number of lattice variables $\{q_j\}_{j=1}^n$ so that we can appeal to Theorem 1.1.

There are still problems in applying Theorem 1.1. In the first place, we do not know that $\ln \rho_\delta$ is in $C^2(\mathbb{R}^n)$; specifically, for scalar Y$_2$, we do not know whether the operator $(1 - K_\delta)^{-1}$, which arises in the computation of $\partial^2 \ln \rho_\delta / \partial q_i \, \partial q_j$, exists in any reasonable sense. Second, when $m_f > 0$, the field $B(z)$ in (1.22) does not go to 0 as $|z| \to \infty$, as is required in Vekua's theory. We overcome both of these problems by imposing a second major cutoff, namely a spatial cutoff on the Fermi mass: we replace m_f in (1.12) and (1.22) by $m_f \chi_{D_R}(x)$, where D_R is the disk centered at 0 with radius $R > 0$. In the case of the Fermi two-point function (1.12) we set

$$S_R = (1 + S_0 m_f \chi_{D_R})^{-1} S_0$$

where S_0 is defined by (1.12) with $m = 0$. We then define $K_R = S_R \phi_\delta \chi_\Lambda$ and

$$\rho_{\delta,R} = c_{\delta,R} \det{}_3(1 - \zeta_R K_R) \exp\{-\tfrac{1}{2}{:}\mathrm{Tr}[(\zeta_R K_R)^2 + K^\dagger K]{:}\} \tag{1.27}$$

where $c_{\delta,R}$ is a normalizing constant and where the additional cutoff $\zeta_R \in C_0^\infty(D_R)$ is introduced for technical reasons. If we assume that $\Lambda \subset D_{R/4}$ and that $\zeta_R = 1$ on $D_{R/2}$, then (1.27) is actually (formally) independent of ζ_R. We are then able to verify condition (1.5) for $\rho_{\delta,R}$ (Corollary 6.6):

$$\partial^2 \ln \rho_{\delta,R} / \partial q_i \, \partial q_j \geqslant 0, \qquad i \neq j \tag{1.28}$$

Taking into account the known ferromagnetic nature of the free boson measure, we may apply Theorem 1.1 to obtain

$$\int fg\rho_{\delta,R} \, d\mu \geqslant \int f\rho_{\delta,R} \, d\mu \int g\rho_{\delta,R} \, d\mu \tag{1.29}$$

for f and g increasing functions of the q_j.

We next wish to take the limit $R \to \infty$. It is possible to let $R \to \infty$ in (1.28) (see Theorem 5.1), but the result is inadequate for our purposes since we obtain the inequality $\partial^2 \ln \rho_\delta / \partial q_i \, \partial q_j \geqslant 0$ for $i \neq j$ only when $(1 - K_\delta)^{-1}$ exists as a bounded operator on $\mathscr{H}$. We are accordingly obliged to take $R \to \infty$

in (1.29), i.e., in the conclusion of Theorem 1.1 rather than in its hypothesis. This involves the proof that (see Section 5)

$$\rho_{\delta,R} \to \rho_\delta \quad \text{in} \quad L^p(d\mu) \tag{1.30}$$

for any $p < \infty$, at least for a subsequence $R_n \to \infty$. The inequality (1.29) now extends to the noncutoff case much as in the $P(\phi)_2$ case (Theorem V.10 of Ref. 5):

Proof of Theorem 1.2. We consider the case of scalar Y_2 with $m_f > 0$. The cases of scalar Y_2 with $m_f = 0$ and pseudoscalar Y_2 with $m_f = 0$ are examined in Sections 7 and 8, respectively. It suffices to consider bounded F and G since we can truncate general F and G and recover (1.23) by limits. Let

$$\phi_\delta(h) = \int \phi_\delta(x)h(x)\,dx = \sum_j \int_{\Delta_j} h(x)\,dx\,q_j$$

Then $F_\delta \equiv F(\phi_\delta(h_1),..., \phi_\delta(h_r)) \to F$ in each $L^p(d\mu)$ as $\delta \to 0$. Hence by (1.26) and (1.29) it suffices to show that

$$\int F_\delta G_\delta \rho_{\delta,R}\,d\mu \geqslant \int F_\delta \rho_{\delta,R}\,d\mu \int G_\delta \rho_{\delta,R}\,d\mu \tag{1.31}$$

But F_δ and G_δ are increasing functions of the q_j since each $\phi_\delta(h_i)$ is. Therefore (1.31) follows from (1.29). ■

We wish to comment on the relation of our lattice approximation to that of MacDermot.[13] As can be seen from (1.25), we impose a lattice cutoff on the boson field but *leave the "Fermi field" untouched*, i.e., we retain the Fermi two-point function $S(x, y)$ of (1.12). MacDermot's lattice spin system differs from ours in that he replaces $S(x, y)$ by its lattice analog as well. But such an approximation would obviously ruin the PDE approach we have taken. MacDermot apparently believed (p. 7 of Ref. 13) that an approximation of our type was not possible, but he did not have any compelling reason to overcome the technical obstacles. His main purpose was to prove the first Griffiths inequality. Actually, it is possible to prove the first Griffiths inequality without resorting to any kind of lattice approximation at all.[15]

Although we find the results of this paper an encouraging step in the search for correlation inequalities for models involving fermions, there remain a number of unanswered questions. As we have already emphasized, our results apply to pseudoscalar Y_2 only in the case $m_f = 0$. However, the calculation of some special cases with $m_f > 0$ (see Section 8) as well as an analysis of pseudoscalar Y_1 lead us to believe that the FKG inequality may hold when $m_f > 0$. Also, one naturally wonders how much of our PDE approach can be extended to higher dimensions. Certainly generalized analytic function theory is peculiar to two dimensions, but we have some evidence that the key inequality (1.19)

$$\operatorname{tr} S'(x, y)S'(x, y) \leqslant 0, \qquad x \neq y$$

holds in higher dimensions (see Section 8). The truth or falsity of this inequality is purely a question about first-order elliptic PDEs that we wish to bring to the attention of workers in that field.

2. L^p ESTIMATES IN THE YUKAWA$_2$ MODEL

The starting point for a rigorous analysis of the Y_2 model is the expression $d\nu = \text{const}\ \rho\ d\mu$ for the measure "after the fermions have been integrated out" [see definitions (1.9)–(1.16)]. Admittedly in the definition

$$B(K) = \tfrac{1}{2}{:}\operatorname{Tr}(K^2 + K^\dagger K){:} \tag{2.1}$$

the counterterm

$$:\operatorname{Tr} K^\dagger K: = \frac{2}{(2\pi)^4}\int \frac{dp}{p^2 + m^2}\int_\Lambda :\phi(x)^2:\, dx \tag{2.2}$$

is infinite and the right side of (2.1) is only formal, but one can give a well-defined meaning to $B(K)$ by defining it as the $L^2(d\mu)$ limit of a suitably regularized version of (2.1); see Ref. 16 or Ref. 10, §VII. We shall always interpret formal objects in such a way in the following analysis.

Seiler's main result[16] for ρ states that

$$\rho \in L^p(d\mu) \quad \text{if} \quad p < \infty \tag{2.3}$$

This result has been generalized in a number of ways, e.g., to include the factor arising from a product of Fermi fields,[16] to include an arbitrary finite boson mass renormalization,[27,28] to give the correct volume dependence,[29,30] and to cover the case of Dirichlet boundary conditions.[10] But for the present purposes we wish to formulate an abstract version of Seiler's theorem that isolates the bounds on K that are sufficient to guarantee (2.3). This will enable us to replace the K of (1.11) with K's involving a lattice cutoff on the boson field and a mass cutoff in the Fermi two point function (1.12).

Given a boson covariance C, a (bounded, real) positive operator on $L^2(\mathbb{R}^2)$, let H_b be the (real) Hilbert space completion of $L^2(\mathbb{R}^2)$ in the inner product $(f, Cg)_{L^2}$. Let $\phi(x)$ be the Gaussian random field with covariance C and mean 0, realized on a probability space $(Q, d\mu)$ (see, e.g., Ref. 22). Let $\mathscr{H}$ be the Fermi Hilbert space as defined in (1.10). Let $\mathscr{C}_p$ denote the class of compact operators A on $\mathscr{H}$ with $\|A\|_p^p \equiv \mathrm{Tr}(A^*A)^{p/2} < \infty$ and let $\mathscr{C}_{p,q}$ denote the class of $\mathscr{C}_p$-valued functions $A(\phi)$ with

$$\|A\|_{p,q} \equiv \left[\int d\mu \|A(\phi)\|_p^q\right]^{1/q} < \infty \tag{2.4}$$

We take $K(\phi)$ to be an integral operator on $\mathscr{H}$ of the form

$$(Kf)_\alpha(x) = (S\chi_\Lambda \phi f)_\alpha(x) \equiv \sum_\beta \int_\Lambda dy\, S_{\alpha\beta}(x, y)\phi(y) f_\beta(y) \tag{2.5}$$

where $S_{\alpha\beta}(x, y)$ is a real, measurable function of x and y for $\alpha, \beta = 0, 1$.

Let $\{\phi_j(x)\}$, $j = 1, 2, \ldots$, be a sequence of Gaussian fields on $(Q, d\mu)$ [thought of as approximations to $\phi(x)$]. For convenience we shall assume that if $i \leqslant j$

$$\int \phi_i(x)\phi(y)\, d\mu = \int \phi_i(x)\phi_j(y)\, d\mu = \int \phi_i(x)\phi_i(y)\, d\mu \equiv C_i(x, y) \tag{2.6a}$$

and that as operators on $L^2(\mathbb{R}^2)$

$$C_i \leqslant C_j \leqslant C \tag{2.6b}$$

Define the "cutoff" object

$$K_j = S\chi_\Lambda \phi_j \tag{2.7a}$$

and the "tails"

$$\delta K_j = K - K_j, \qquad \delta B_j = :\mathrm{Tr}(K + K^\dagger)\, \delta K_j: \tag{2.7b}$$

Let D be a self-adjoint operator in $\mathscr{H}$ with bounded inverse D^{-1}. [In practice, $D = (-\Delta + m_f^2)^{1/2}$.] Define (formally)

$$W(K) = \tfrac{1}{2}:\mathrm{Tr}(K^\dagger K - K^* K): = \int w(x, y) :\phi(x)\phi(y): dx\, dy \tag{2.8}$$

Then we have the following theorem (where the letters c and d denote various positive constants independent of the indices i and j; each c does not necessarily represent the same constant).

Theorem 2.1. Assume that for some $\epsilon > 0$ the following conditions are satisfied:

(i) $\|D^{-2\epsilon}K\|_{2,2} + \|D^\epsilon K\|_{4,4} + \|B\|_{L^2} < \infty$.
(ii) $\|D^{-2\epsilon}\delta K_j\|_{2,2} + \|D^\epsilon \delta K_j\|_{4,4} + \|\delta B_j\|_{L^2} \leqslant c \exp(-cj^d)$.
(iii) $\int d\mu\, \mathrm{Tr}(K_j + K_j^*)^2 \leqslant cj$.
(iv) $\|W\|_{L^2} = \sqrt{2}\, \|w\|_{H_b \otimes H_b} < \infty$.
(v) $w(x, y)$ defines a positive-definite quadratic form on $C_0^\infty \times C_0^\infty$.

Then $\rho(K) \in L^p(d\mu)$ for any $p < \infty$. Moreover, if the estimates in (i)–(iv) hold uniformly in some parameter, then $\|\rho\|_{L^p}$ is bounded uniformly.

Remark. We pause to explain how the hypotheses ensure that the various objects that occur are well-defined. By complex interpolation[28]

$$\|K\|_3 \leqslant \|D^\epsilon K\|_4^{2/3} \|D^{-2\epsilon} K\|_2^{1/3} \tag{2.9}$$

and so by (i), $K \in \mathscr{C}_{3,3}$; hence $K \in \mathscr{C}_3$ a.e. and $\det_3(1 - \lambda K)$ is a well-defined function on $(Q, d\mu)$ (Ref. 24, p. 1106).

Of course $W(K)$ and $B(K)$ are to be interpreted as limits in $L^2(d\mu)$. The equality in (iv) is standard:

$$\begin{aligned} \|W\|_{L^2}^2 &= \int w(x, y) w(x', y')[C(x, x')C(y, y') + C(x, y')C(y, x')] \\ &= 2\|w\|_{H_b \otimes H_b}^2 \end{aligned}$$

If we let H_1 be the Hilbert space completion of C_0^∞ with norm $\|f\|_{H_1} = \|C^{-1/2} f\|_{L^2}$, then

$$(f, W_{\text{op}} g)_{L^2} \equiv \int f(x) w(x, y) g(y)\, dx\, dy = (f, C W_{\text{op}} g)_{H_1}$$

Estimate (iv) then says that the integral operator CW_{op} on H_1 is Hilbert–Schmidt, for

$$\|CW_{\text{op}}\|_{\text{HS}}^2 = \|w\|_{H_b \otimes H_b}^2 < \infty$$

The proof below involves operators of the form

$$T(\lambda) = \wedge^n (1 - \lambda K)^{-1} \det_3(1 - \lambda K)$$

on the antisymmetric product space $\wedge^n \mathscr{H}$. We do not know whether $R(\lambda) = (1 - \lambda K)^{-1}$ is well-defined as an element of $\mathscr{L}(\mathscr{H})$ a.e. in ϕ. However, there is a cancellation between poles of $R(\lambda)$ and zeros of $\det_3(1 - \lambda K)$; more precisely, by Ref. 28, Prop. 5, for $n \geqslant 0$

$$K \mapsto \wedge^n (1 - K)^{-1} \det_3(1 - K) \tag{2.10}$$

is a continuous map from $\mathscr{C}_3$ to $\mathscr{L}(\wedge^n \mathscr{H})$. But $K \in \mathscr{C}_3$ a.e., so we can define $T(\lambda)$ a.e. in ϕ by continuity in λ even at values $\lambda^{-1} \in \sigma(K)$. Since the "operator" R will always occur in such a combination, we shall manipulate with K and R as though they were well-defined, bounded operators for all ϕ and λ. For notational simplicity we now set $\lambda = 1$.

Proof of Theorem 2.1. We present the bare bones of the proof, referring the reader to Ref. 10 for some missing details. The basic idea is to obtain an expansion for $\rho(K)$ in which the jth term depends (essentially) on K_j rather than K. A related proof may be found in McBryan[29] and the basic method goes back to a paper of Glimm and Jaffe.[31]

Obviously,

$$\rho(K) - \rho(K_1) = \int_0^1 ds_1 \frac{d}{ds_1} \rho(\tilde{K}_1) \tag{2.11}$$

where

$$\tilde{K}_1(s_1) = (1 - s_1)K_1 + s_1 K$$

We compute that (see Ref. 10, §III)

$$\frac{d}{ds_1} \rho(\tilde{K}_1) = -[\mathrm{Tr}(\tilde{R}_1 A_1) + B_1]\rho(\tilde{K}_1)$$

where

$$\tilde{R}_1 = (1 - \tilde{K}_1)^{-1}, \qquad A_1 = \tilde{K}_1{}^2\, \delta K_1, \qquad B_1 = :\mathrm{Tr}[(\tilde{K}_1 + \tilde{K}_1{}^\dagger)\, \delta K_1]:$$

Thus from (2.11)

$$\rho = \rho(K_1) - \int_0^1 ds_1\, [\mathrm{Tr}(\tilde{R}_1 A_1) + B_1]\rho(\tilde{K}_1) \tag{2.12}$$

The term $\rho(K_1)$ on the right of (2.12) is the first-order term in our expansion. To obtain the second-order terms we iterate the procedure (2.11): we replace $\tilde{K}_1$ and $\tilde{R}_1$ in (2.12) by

$$\hat{K}_2(s_1) = (1 - s_1)K_1 + s_1 K_2, \qquad \hat{R}_2(s_1) = [1 - K_2(s_1)]^{-1}$$

by interpolating via

$$\tilde{K}_2(s_1, s_2) = (1 - s_1)K_1 + (s_1 - s_2)K_2 + s_2 K, \qquad 0 \leqslant s_2 \leqslant s_1$$
$$\tilde{R}_2 = (1 - \tilde{K}_2)^{-1}$$

(A note on notation: $\hat{K}_j$ will be a convex combination of $K_1, K_2, \ldots, K_j$, whereas $\tilde{K}_j$ will be a convex combination of $K_1, \ldots, K_j$ *and* K.) For example,

$$\mathrm{Tr}(\tilde{R}_1 A_1)\rho(\tilde{K}_1) = \mathrm{Tr}(\hat{R}_2 A_1)\rho(\hat{K}_2) + \int ds_2 \frac{d}{ds_2} \mathrm{Tr}(\tilde{R}_2 A_1)\rho(\tilde{K}_2)$$

The computation of the s_2 derivative involves a certain amount of algebra (see Ref. 10, §III) as well as the introduction of some additional structure: let $T_n(\cdot) = n!\, \mathrm{Tr}_{\wedge^n \mathscr{H}}(\cdot)$. Then

$$\frac{d}{ds_2} \mathrm{Tr}(\tilde{R}_2 A_1)\rho(\tilde{K}_2) = [-T_2(\wedge^2 \tilde{R}_2 \cdot A_1 \wedge A_2) + T_1(\tilde{R}_2 A_1 E_2) - T_1(\tilde{R}_2 A_1) B_2]\rho(\tilde{K}_2) \tag{2.13}$$

where

$$A_j = \tilde{K}_j^2 \, \delta K_j \tag{2.14a}$$

$$E_j = (1 + \tilde{K}_j) \, \delta K_j \tag{2.14b}$$

$$B_j = :\mathrm{Tr}(\tilde{K}_j + \tilde{K}_j^\dagger) \, \delta K_j: \tag{2.14c}$$

The virtues of formula (2.13) are that (a) $(1 - \tilde{K}_2)^{-1}$ appears everywhere as in (2.10), so that we can obtain the pole-zero cancellation; and (b) (2.13) can be iterated easily. For example, the s-derivative of the term $T_n(\wedge^n \tilde{R}_{n+1} A_1 \wedge \cdots \wedge A_n) \quad (\tilde{K}_{n+1})$ is

$$[-T_{n+1}(\wedge^{n+1}\tilde{R}_{n+1}A_1 \wedge \cdots \wedge A_{n+1}) + \sum_{j=1} T_n(\wedge^n \tilde{R}_{n+1} A_1 \wedge \cdots \wedge A_j E_{n+1} \wedge \cdots \wedge A_n) - T_n(\wedge^n \tilde{R}_{n+1} A_1 \wedge \cdots \wedge A_n) B_{n+1}] \rho(\tilde{K}_{n+1})$$

For details see Ref. 10.

The final expansion has the form

$$\rho = \sum_{m=1}^{\infty} \sum_{\alpha} \rho_m^{(\alpha)} \tag{2.15}$$

where at the mth order we get a sum $\sum_\alpha \rho_m^{(\alpha)}$ of at most $m!$ terms each of the form

$$\rho_m^{(\alpha)} = \int_{0 \leq s_{m-1} \ldots \leq s_2 \leq s_1 \leq 1} \cdots \int ds_1 \cdots ds_{m-1} \, T_r(\wedge^r \hat{R}_m G_1 \wedge \cdots \wedge G_r) \prod_k B_{i_k} \rho(\hat{K}_m) \tag{2.16}$$

where $\hat{K}_m$ is an s-dependent, convex combination of $K_1, \ldots, K_m$; $\hat{R}_m = (1 - \hat{K}_m)^{-1}$; G_j has the form $A_i \prod_k E_{i_k}$ with A_j, E_j, B_j defined in (2.14); and

$$0 \leqslant r \leqslant m - 1 \tag{2.17}$$

Now if $\rho_m^{(\beta)}$ is that $\rho_m^{(\alpha)}$ with the largest L^p norm, we have from (2.15) that

$$\|\rho\|_{L^p} \leqslant \sum_m m! \, \|\rho_m^{(\beta)}\|_{L^p} \tag{2.18}$$

To estimate $|\rho_m^{(\beta)}|$ we appeal to Lemma A.2 of Ref. 10 to obtain

$$\begin{aligned} |\rho_m{}^\beta| &\leqslant r! \, \|\wedge^r \hat{R}_m\| \cdot \|G_1 \wedge \cdots \wedge G_r\|_1 \prod_k |B_{i_k}| \, |\rho(\hat{K}_m)| \\ &\leqslant \left(\prod |B_{i_k}|\right)\left(\prod \|G_j\|_1\right)(\|\wedge^r \hat{R}_m\| \cdot |\rho(\hat{K}_m)|) \end{aligned} \tag{2.19}$$

The cancellation of poles against zeros is provided by the estimate [recall the definition (1.9) of ρ]

$$\|\wedge^r \hat{R}_m\| \cdot |\det_3(1 - \hat{K}_m)| \leqslant \exp[r/2 + \mathrm{Tr}(\hat{K}_m + \hat{K}_m{}^*)^2/4] \tag{2.20}$$

As pointed out by McBryan (Ref. 27, Lemma 33) this estimate is an elementary generalization of Carleman's inequality.[24] Now

$$B(K) = \tfrac{1}{4}{:}\mathrm{Tr}(K + K^*)^2{:} + \tfrac{1}{2}{:}\mathrm{Tr}(K^\dagger K - K^* K){:}$$
$$= \tfrac{1}{4}\,\mathrm{Tr}(K + K^*)^2 - \tfrac{1}{4}\int d\mu\, \mathrm{Tr}(K + K^*)^2 + W(K)$$

It follows from (2.20) that

$$\| \wedge^r \hat{R}_m\|\, |\rho(\hat{K}_m)| \leqslant \exp\left[r/2 + \int d\mu\, \mathrm{Tr}(\hat{K}_m + \hat{K}_m{}^*)^2/4 - W(\hat{K}_m)\right] \tag{2.21a}$$

Now, to estimate $\int d\mu\, \mathrm{Tr}(\hat{K}_m + \hat{K}_m{}^*)^2$, we recall that $\hat{K}_m$ is an s-dependent, convex combination of $K_1,\ldots, K_m$:

$$\hat{K}_m = \sum_{j=1}^{m} \lambda_j K_j, \qquad \lambda_j \geqslant 0, \qquad \sum_{j=1}^{m} \lambda_j = 1$$

We have

$$\int d\mu\, \mathrm{Tr}(K_m + K_m{}^*)^2 = \sum_{j,j'=1}^{m} \lambda_j\lambda_{j'} \int \mathrm{Tr}(K_j + K_j{}^*)(K_{j'} + K_{j'}^*)\, d\mu$$
$$= \sum_{j,j'=1}^{m} \lambda_j\lambda_{j'} \int d^2x \int d^2y\, v(x, y) \int \phi_j(x)\phi_{j'}(y)\, d\mu$$

where v is the configuration kernel of $\mathrm{Tr}(K + K^*)^2$. It follows from (2.6a) and hypothesis (iii) that

$$\int d\mu\, \mathrm{Tr}(\hat{K}_m + \hat{K}_m{}^*)^2 = \sum_{j,j'=1}^{m} \lambda_j\lambda_{j'} \int d\mu\, \mathrm{Tr}(K_{\min\{j,j'\}} + K_{\min\{j,j'\}}^*)^2$$
$$\leqslant \sum_{j,j'=1}^{m} \lambda_j\lambda_{j'} c \min\{j, j'\}$$
$$\leqslant cm$$

Thus, by (2.19), (2.21a), and (2.17)

$$\|\rho_m{}^\beta\|_{L^p} \leqslant [\exp(cm)] \left\| \prod B_{i_k} \prod \|G_j\|_1 \exp[-W(\hat{K}_m)] \right\|_{L^p}$$
$$\leqslant [\exp(cm)] \left\| \prod B_{i_k} \prod \|G_j\|_1 \right\|_{L^{2p}} \|\exp[-W(\hat{K}_m)]\|_{L^{2p}} \tag{2.21b}$$

by the Schwarz inequality. We estimate each G_j by

$$\|G_j\|_1 = \left\| A_i \prod E_{i_k} \right\|_1 \leqslant \|A_i\|_1 \prod \|E_{i_k}\|$$

There are exactly $m - 1$ of the A_i, B_i, and E_i occurring in (2.21b), so that by the Hölder inequality with $q = 2p(m - 1)$

$$\left\| \prod B_{i_k} \prod \|G_j\|_1 \right\|_{L^{2p}} \leqslant \prod \|B_{i_1}\|_{L^q}\| \; \|A_{i_2}\|_1\|_{L^q}\| \; \|E_{i_3}\| \; \|_{L^q} \qquad (2.22)$$

the latter product occurring over $i_k \in I_k$, where $\{I_1, I_2, I_3\}$ is a partition of $\{1, 2,..., m - 1\}$.

We estimate each $\|A_j\|_1$ factor by Hölder's inequality and (2.9); i.e., from (2.14a)

$$\begin{aligned} \|A_j\|_1 &\leqslant \|\tilde{K}_j\|_3{}^2 \|\delta K_j\|_3 \\ &\leqslant \|D^\epsilon \tilde{K}_j\|_4^{4/3} \|D^{-2\epsilon}\tilde{K}_j\|_2^{2/3} \|D^\epsilon \delta K_j\|_4^{2/3} \|D^{-2\epsilon}\delta K_j\|_2^{1/3} \end{aligned}$$

By (i), $D^{-2\epsilon}\delta K_j = D^{-2\epsilon}K - K^{-2\epsilon}K_j$ is in $\mathscr{C}_{2,2}$ uniformly in j and hence so is $D^{-2\epsilon}\tilde{K}_j$. In this way we deduce from hypotheses (i) and (ii) that

$$\| \; \|A_j\|_1 \; \|_{L^2} \leqslant c \exp(-cj^d)$$

But $\|A_j\|_1$ is cubic in ϕ, so that by hypercontractivity (Ref. 22, §1.5) we actually have

$$\| \; \|A_j\|_1 \; \|_{L^q} \leqslant c(q - 1)^{3/2} \exp(-cj^d)$$

where the constants are independent of j and q. We estimate the E_j factors in (2.22) by similar reasoning from

$$\|E_j\| = \|(1 + \tilde{K}_j)\, \delta K_j\| \leqslant \|\delta K_j\|_4 + \|\tilde{K}_j\|_4 \|\delta K_j\|_4$$

The desired bound on B_j follows from

$$\|B_j\|_{L^2}^2 \leqslant \|\delta B_j\|_{L^2}^2 \qquad (2.23)$$

and hypothesis (ii). The bound (2.23) is almost obvious: the only difference between δB_j and B_j [see definitions (2.7b) and (2.14c)] is that the K in δB_j is replaced by $\tilde{K}_j$ in B_j, where $\tilde{K}_j$ is a convex combination of K_1,..., K_j, K. If one writes out both sides of (2.23), then (2.23) follows by appropriate use of (2.6).

The upshot is that (2.22) is bounded by [with $q = 2p(m - 1)$]

$$\begin{aligned} \prod_{j=1}^{m-1} c(q - 1)^{3/2} \exp(-cj^d) &\leqslant c^m m^{3m/2} \prod_{j=1}^{m-1} \exp(-cj^d) \\ &\leqslant c^m m^{3m/2} \exp(-cm^{1+d}) \end{aligned}$$

The point is that the decay factors $\exp(-cj^d)$ accumulate to give an overall convergence factor $\exp(-cm^{1+d})$ that dominates m^m. Returning to (2.21), we conclude that

$$\|\rho_m{}^\beta\|_{L^p} \leqslant c^{m \log m - cm^{1+d}} \|\exp[-W(\hat{K}_m)]\|_{L^{2p}} \tag{2.24}$$

By conditioning (Ref. 32, §III, 2) and (2.6b) we have

$$\|\exp[-W(\hat{K}_m)]\|_{L^{2p}} \leqslant \|\exp[-W(K)]\|_{L^{2p}}$$

Since CW_{op} is a *positive* HS operator on H_1, explicit integration (see, e.g., Ref. 16, Lemma 3.3) gives

$$\|\exp[-W(K)]\|_{L^{2p}} \leqslant \exp[2p\|W\|_{L^2}^2] < \infty$$

Combining (2.18) and (2.24), we get

$$\|\rho\|_{L^p} \leqslant \sum_m m!\, c^{m \log m - cm^{1+d}} < \infty$$

Clearly this bound will be uniform in any parameters if each of the input bounds is. ■

In the usual spatially cutoff Y_2 model we take

$$C = (-\Delta + m_b{}^2)^{-1}, \qquad m_b > 0 \tag{2.25a}$$

$$D = (-\Delta + m_f{}^2)^{1/2} \tag{2.25b}$$

$$S(x, y) \equiv (-\not{p} + m_f)^{-1}(x, y) \tag{2.25c}$$

and

$$\phi_j(x) = h_{\kappa_j} * \phi(x) \tag{2.25d}$$

where $\hat{h}_\kappa(k) = (2\pi)^{-2}\chi_\kappa(k)$, with χ_κ the characteristic function of the set $|k| \leqslant \kappa$ and with $\kappa_j = e^j$. Then

$$C_j(x, y) = \frac{1}{(2\pi)^2}\int_{|k| \leq \kappa_j} \frac{e^{ik(x-y)}}{k^2 + m_b{}^2}\, dk$$

and (2.6a) and (2.6b) are immediate. The hypotheses (i)–(vi) of Theorem 2.1 may be verified by direct computation (see Seiler[16]). An important feature of these computations is the explicit momentum space cancellations in expressions involving Tr K^2, Tr K^*K, and Tr $K^\dagger K$. Since we shall make frequent use of these objects (or variations of them) we record here (formal) momentum space integrals for their kernels: Tr K^2 has the form

$$\text{Tr}\, K^2 = \int_\Lambda \int_\Lambda w_{K^2}(x - y)\phi(x)\phi(y)$$

where

$$\hat{w}_{K^2}(k) = \frac{2}{(2\pi)^3}\int \frac{m_f{}^2 - p\cdot(p + k)}{D(p)^2 D(p + k)^2} \tag{2.26a}$$

Tr K^*K and Tr $K^\dagger K$ have a similar form with

$$\hat{w}_{K^*K}(k) = \frac{2}{(2\pi)^3}\int \frac{1}{D(p)D(p+k)}\,dp \tag{2.26b}$$

and

$$\hat{w}_{K^\dagger K}(k) = \frac{2}{(2\pi)^3}\int \frac{1}{D(p)^2}\,dp \tag{2.26c}$$

For example, in (2.8) the kernel $w(x, y) = w_0(x-y)\chi_\Lambda(x)\chi_\Lambda(y)$, where

$$\hat{w}_0(k) = \frac{2}{(2\pi)^3}\int [D(p)^{-2} - D(p+k)^{-1}D(p)^{-1}]\,dp$$

Hypothesis (v) then follows from the inequality

$$\hat{w}_0(k) \geqslant 0$$

which in turn follows from the elementary inequality

$$D(p+k)^{-1}D(p)^{-1} \leqslant \tfrac{1}{2}[D(p+k)^{-2} + D(p)^{-2}]$$

In addition, we wish to establish L^p bounds in a case where the operator S of (2.5) is *not* diagonal in momentum space (see Section 5). To this end we follow Seiler and Simon[28,30] (see also McBryan[27,29] and Cooper and Rosen[10]) in estimating the "low-momentum" and "high-momentum" parts of K separately. For $K = S\chi_\Lambda\phi$ with S given by (2.25c) we write $K = L + H$, where $L = S_\sigma\chi_\Lambda\phi$ with

$$S_\sigma(x, y) = \frac{1}{(2\pi)^2}\int_{|p|\leq\sigma} (-\not{p} + m_f)^{-1}e^{ip(x-y)}\,dp$$

and $\sigma > 0$ a constant to be determined. It is not hard to see that (Ref. 28, Lemma 2.5) there is an operator T linear in ϕ such that

$$\|L\|_1 \leqslant \|T\|_2 \qquad \text{with } T \in \mathscr{C}_{2,2} \tag{2.27a}$$

(Explicitly, $T = \text{const } D^{-2}\chi_\Lambda\phi$.) This estimate allows us to replace K by H at certain points in the estimation of ρ [see (2.36) below]. The key idea is that if we replace K^*K by H^*H in the definition (2.8) of W we improve the positivity property of w. Explicitly, if

$$W_H = \tfrac{1}{2}{:}\mathrm{Tr}(K^\dagger K - H^*H){:} \equiv \int_{\Lambda\times\Lambda} w_H(x, y){:}\phi(x)\phi(y){:}\,dx\,dy \tag{2.27b}$$

then the kernel $w_H(x, y) = w_H(x - y)$ has Fourier transform [see (2.26)]

$$\hat{w}_H(k) = \frac{2}{(2\pi)^3}\int [D(p)^{-2} - \theta_\sigma(p+k)D(p+k)^{-1}D(p)^{-1}]\,dp \tag{2.28}$$

where $\theta_\sigma(p)$ is the characteristic function of the set $\{p| \ |p| \geqslant \sigma\}$. By the same reasoning that led to (2.26), one obtains[28]

$$\hat{w}_H(K) \geqslant (1/4\pi^2) \ln(1 + \sigma^2/m_f^2) \tag{2.29}$$

which can be made arbitrarily large by a suitable choice of σ. Thus the Gaussian e^{-W_H}, rather than being merely integrable, can be exploited to dominate other Gaussian factors. These remarks motivate the following perturbation theorem, which may be regarded as an abstraction of results of McBryan[27] and Seiler and Simon.[28,30]

Theorem 2.2. Let K and $K' = K + V$ be operators of the form (2.5) and define

$$\rho(K', K) = \det_3(1 - \lambda K') \exp[-\lambda^2 B(K', K)] \tag{2.30a}$$

where

$$B(K', K) = \tfrac{1}{2}{:}\mathrm{Tr}[(K')^2 + K^\dagger K]{:} \tag{2.30b}$$

As in (2.7), define the tails δK_j and $\partial K_j'$ and define

$$\delta B_j(K', K) = {:}\mathrm{Tr}(K' \, \delta K_j' + K^\dagger \, \delta K_j){:} \tag{2.30c}$$

Suppose that

(a) K satisfies the hypotheses (i) and (iii) of Theorem 2.1.

(b) For any $M > 0$, K can be written as a sum of operators linear in ϕ, $K = L + H$, such that L satisfies (2.27a) for some T, the kernel W_H in (2.27b) defines a quadratic form on $C_0^\infty \times C_0^\infty$ bounded below by M, and

$$\|\chi_\Lambda w_H \chi_\Lambda\|_{H_b \otimes H_b} < \infty \tag{2.31}$$

(c) The tails $\delta K_j'$ and δB_j satisfy the decay bounds (ii) of Theorem 2.1.

(d) With the $\epsilon > 0$ specified by Theorem 2.1,

$$\|D^{2\epsilon} V\|_{2,2} + \|V D^{2\epsilon}\|_{2,2} < \infty \tag{2.32}$$

Then, if $p < \infty$, $\rho(K', K) \in L^p(d\mu)$, uniformly in parameters if the hypotheses hold uniformly.

Proof. We apply the same expansion procedure to $\rho(K', K)$ as in Theorem 2.1. The proof is identical up to (2.19). Moreover, there is no change in the estimation of the B_j and G_j factors in (2.19) since by (a) and (d), $K' = K + V$ satisfies

$$\|D^{-2\epsilon} K'\|_{2,2} + \|D^\epsilon K'\|_{4,4} < \infty \tag{2.33}$$

obviously, and

$$\|B(K', K)\|_{L^2} < \infty \tag{2.34}$$

(2.34) follows from (2.32) since $B(K) \in L^2$ by (a) and

$$\begin{aligned} B(K', K) - B(K) &= \tfrac{1}{2}{:}\mathrm{Tr}[(K')^2 - K^2]{:} \\ &= \tfrac{1}{2}{:}\mathrm{Tr}\, V(K' + K){:} \\ &= \tfrac{1}{2}\,\mathrm{Tr}\, V(K' + K) - \tfrac{1}{2}\int d\mu\, \mathrm{Tr}\, V(K' + K) \end{aligned}$$

But by the Hölder inequality

$$|\mathrm{Tr}\, VK| = |\mathrm{Tr}\, VD^{2\epsilon}D^{-2\epsilon}K| \leqslant \|VD^{2\epsilon}\|_2 \|D^{-2\epsilon}K\|_2 \tag{2.35}$$

so that $\mathrm{Tr}\, VK \in L^p(d\mu)$ for any $p < \infty$, and similarly for $\mathrm{Tr}\, VK'$ by (2.33).

It thus remains to bound the last factor in (2.19), namely $\|\wedge^r \hat{R}_m'\|\,|\rho(\hat{K}_m')|$, where $\hat{K}_m'$ is a convex combination of $K_1',\ldots, K_m'$ and $\hat{R}_m' = (1 - \hat{K}_m')^{-1}$. (For notational convenience we shall drop the carets because in the end the occurrence of convex combinations is handled as in Theorem 2.1.) The bound is based on this generalization of (2.20):

$$\|\wedge^r R\|\,|\mathrm{det}_3(1 - K)| \leqslant \exp[cr + c\|L\|_1 + \tfrac{1}{2}\,\mathrm{Tr}(K^2 + H^*H)] \tag{2.36}$$

where $K = L + H$. As noted in Ref. 10, §VII, (2.36) is a simple consequence of Lemma 2.3 of Seiler and Simon.[28]

We define H' by $K' = K + H'$ (and H_m' by $K_m' = L_m + H_m'$), so that

$$V = K' - K = H' - H \tag{2.37}$$

Applying (2.36), we obtain

$$\begin{aligned} &\|\wedge^r R_m'\|\,|\rho(K_m')| \\ &\quad\leqslant \exp\{cr + c\|L_m\|_1 + \tfrac{1}{2}\,\mathrm{Tr}[(K_m')^2 + (H_m')^*H_m'] - B(K_m', K_m)\} \\ &\quad= \exp\{cr + c\|L\|_1 + \tfrac{1}{2}{:}\mathrm{Tr}[(H_m')^*H_m' - K_m^{\dagger}K_m]{:} \\ &\qquad + \tfrac{1}{2}\int d\mu\, \mathrm{Tr}[(K_m')^2 + (H_m')^*H_m']\} \end{aligned} \tag{2.38}$$

If we substitute $H_m' = H_m + V_m$, the first trace in (2.38) can be written as

$$\begin{aligned} &\tfrac{1}{2}{:}\mathrm{Tr}(H_m^*H_m - K^{\dagger}K_m){:} + \tfrac{1}{2}{:}\mathrm{Tr}(H_m^*V_m + V_m^*H_m + V_m^*V_m){:} \\ &\quad \equiv -W_{H_m} + Q_{1,m} \end{aligned} \tag{2.39}$$

where W_H is defined in (2.27). If we substitute $K_m' = K_m + V_m$ and $H_m' = K_m + V_m - L_m$ into the last term in (2.38), we get

$$\begin{aligned} &\mathrm{Tr}[(K_m')^2 + (H_m')^*H_m'] - \mathrm{Tr}(K_m^2 + K_m^*K_m) \\ &\quad = \mathrm{Tr}[K_mV_m + V_mK_m + V_m^2 + K_m^*(V_m - L_m) \\ &\qquad + (V_m - L_m)K_m^* + (V_m - L_m)^*(V_m - L_m)] \\ &\quad \equiv 2Q_2 \end{aligned}$$

We thus rewrite (2.38) as

$$\|\wedge^r R_m'\| \, |\rho(K_m')| \leqslant \exp\Big[cr + \int d\mu\, Q_2 + \tfrac{1}{2}\int d\mu\, (K_m{}^2 + K_m{}^*K_m)$$
$$+ c\|L_m\|_1 + Q_{1,m} - W_{H_m}\Big]$$

Now $\int d\mu\, Q_2 < \infty$ uniformly in m by the same reasoning as in (2.35); terms involving L_m are no problem since $L \in \mathscr{C}_{1,2}$ by (2.27a). The term $\int d\mu\, (K_m{}^2 + K_m{}^*K_m) \leqslant cm$ by assumption (a). Hence

$$\| \, \|\wedge^r R_m'\| \rho(K_m')\|_{L^p} \leqslant [\exp(cr)] \|\exp(c\|L_m\|_1 + Q_{1,m} - W_{H_m})\|_{L^p} \tag{2.40}$$

By (2.27a)

$$\|L_m\|_1 \leqslant \|T_m\|_2 \leqslant \tfrac{1}{2}(\delta^{-1} + \delta\|T_m\|_2{}^2) \tag{2.41a}$$

for any $\delta > 0$. Now by the assumptions on T

$$\|T_m\|_2{}^2 = \int t(x, y)\phi_m(x)\phi_m(y)\, dx\, dy$$

where t defines a positive-definite operator T_{op} on L^2 with CT_{op} trace class. We write

$$\|T_m\|_2{}^2 = \int t(x, y) {:}\phi_m(x)\phi_m(y){:}\, dx\, dy + \operatorname{tr}(C_m T_{\text{op}}) \tag{2.41b}$$

Since $\operatorname{tr}(C_m T_{\text{op}}) \leqslant \operatorname{tr}(CT_{\text{op}}) < \infty$ we conclude from (2.40), (2.41), and conditioning[32] that

$$\| \, \|\wedge^r R_m'\| \rho(K_m')\|_{L^p} \leqslant \exp(cr + c\delta^{-1}) \|\exp(c\delta\|T\|_2{}^2 + Q_1 - W_H)\|_{L^p}$$

From its definition (2.39) and by reasoning as in (2.35), $Q_1 \in L^2(d\mu)$. For M sufficiently large, it follows from Theorem 3.2 of Ref. 30 that $\exp(Q_1 - W_H) \in L^{2p}$. We then choose $\delta > 0$ sufficiently small so that $\exp(c\delta\|T\|_2{}^2) \in L^{2p}$ (see, e.g., Theorem 3.1 of Ref. 30). The proof of the theorem is completed just as in the case of Theorem 2.1. ■

We conclude this section with a continuity result for $\rho(K)$ as a function of K. First we quote this useful result of Seiler and Simon (Ref. 33, Lemma 3.5):

Lemma 2.3. Let $(Q, d\mu)$ be a probability measure space. Suppose $f_n \in L^p$ with $\sup_n \|f_n\|_{L^p} < \infty$. If $f_n \to f$ pointwise, then $f \in L^p$ and $f_n \to f$ in L^q for any $q < p$.

We then prove:

Lemma 2.4. Let K and K_n, $n = 1, 2, \ldots$, be operators of the form (2.5). Assume that for any $p < \infty$:

(i) $\|\rho(K_n)\|_{L^p} < \infty$ uniformly in n.
(ii) $\|K_n - K\|_{3,3} \to 0$.
(iii) $\|B(K_n) - B(K)\|_{L^2} \to 0$.

Then for any $p < \infty$, $\rho(K) \in L^p$ and for a subsequence $\{K_{n_k}\}$, $\rho(K_{n_k}) \to \rho(K)$ in L^p.

Remark. An analogous result holds for $\rho(K', K)$ of (2.30).

Proof. By (ii) and (iii) there is a subsequence $\{K_{n_k}\}$ such that

$$K_{n_k} \to K \quad \text{in} \quad \mathscr{C}_3 \quad \text{and} \quad B(K_n) \to B(K) \qquad \text{a.e. (w.r.t. } \phi)$$

Using (2.10), we deduce that $\rho(K_{n_k}) \to \rho(K)$ a.e. The conclusion of the lemma then follows from (i) and Lemma 2.3. ■

3. GENERALIZED ANALYTIC FUNCTIONS

Let S_0 be the massless Fermi propagator, i.e.,

$$S_0(x, y) = \frac{1}{(2\pi)^2} \int dp \, \frac{\not{p}}{p^2} \, e^{ip\cdot(x-y)}$$

and let $h(x)$ be a real-valued, bounded, measurable function. In this section S' denotes the operator

$$S' = (1 - S_0 h)^{-1} S_0 \tag{3.1a}$$

on the "massless" Sobolev space $\mathscr{H}_0$ [recall the definition (1.10) of $\mathscr{H}_m$]. What we prove about S' will be applied in Section 5 to the case

$$h(x) = \phi_\delta(x)\chi_\Lambda(x) - m\chi_{D_R}(x) \tag{3.1b}$$

where the lattice cutoff boson field ϕ_δ is a simple function (see Section 4) and D_R is the disk centered at 0 with radius $R > 0$. Here $m\chi_{D_R}$ is the spatially cutoff Fermi mass. This cutoff will be removed in Section 5, but it seems essential for the partial differential equations approach of the present section.

Our main results are:

Theorem 3.1. $S_0 h$ is a $\mathscr{C}_4$ operator on $\mathscr{H}_0$ and $(1 - S_0 h)^{-1}$ exists as a bounded operator on $\mathscr{H}_0$.

Theorem 3.2. The restriction of S' to smooth functions with compact support is an integral operator whose kernel $S'(x, y)$ is locally integrable on $\mathbb{R}^4$ and can be chosen to be continuous in x on $\mathbb{R}^2\backslash\{y\}$ and continuous in y on $\mathbb{R}^2\backslash\{x\}$.

Theorem 3.3. tr $S'(x, y)S'(y, x) < 0$ for $x \neq y$.

Our first step toward proving these theorems is to recast everything in complex notation. For example, if we set $z = x_0 + ix_1$, it is not hard to see that

$$S_0^{00}(x) + iS_0^{10}(x) = -i/\pi z$$
$$S_0^{01}(x) + iS_0^{11}(x) = -1/\pi z$$

so

$$(S_0 v)_0(x) + i(S_0 v)_1(x) = \frac{-i}{\pi}\int d^2\zeta \, \frac{\overline{v(\zeta)}}{z - \zeta}$$

for $v \in C_0^\infty(\mathbb{R}^2)$, where $v = v_0 + iv_1$ is identified with the vector (v_0, v_1). Now for a given two-component function $g \in \mathscr{H}_0$, the function $f = (1 - S_0 h)^{-1}g$ solves the integral equation

$$f - S_0 hf = g \tag{3.2}$$

if it exists, so the basic integral equation we must study is

$$f + Pf = g \tag{3.3}$$

where the (antilinear) operator P is defined by

$$(Pf)(z) = \frac{i}{\pi}\int d^2\zeta \, \frac{1}{z - \zeta} \, h(\zeta)\overline{f(\zeta)}$$

[If the components of f and g are complex, then Eq. (3.2) splits into real and imaginary parts, so there are no difficulties with our complexification procedure.] Since $1/\pi z$ is a fundamental solution for the Cauchy–Riemann operator $\partial_{\bar z} = \frac{1}{2}(\partial/\partial x_0 + i\,\partial/\partial x_1)$ [and in matrix notation this would be the statement that $S_0(x)$ is a fundamental solution for the operator $-\beta_0\,\partial/\partial x_0 - \beta_1\,\partial/\partial x_1$], the differential equation associated to (3.3) is, at least formally,

$$\partial_{\bar z} f + ih\bar f = \partial_{\bar z} g$$

Such generalizations of the Cauchy–Riemann equations are treated extensively in Refs. 25 and 26.

Our second step is to draw on the theory developed in Ref. 25. The homogeneous differential equation of interest to us is

$$\partial_{\bar z} f + B\bar f = 0 \tag{3.4}$$

where B is a bounded, measurable function with compact support. The solutions of such an equation are called *generalized analytic functions* and they are shown by Vekua and Bers to have many of the properties of analytic functions. It should be emphasized here that the conditions placed on the variable coefficient are ours. The results of their theory are proven for a much larger space of variable coefficients.

The first lemma we wish to state is central to the theory of generalized analytic functions. We give only a formal proof because it clearly illustrates the idea behind the technically complicated theory of Vekua and Bers. The ingredients are an imitation of the elementary method of solving a linear ordinary differential equation and the knowledge that entire functions are exactly those functions annihilated by the Cauchy–Riemann operator in the sense of distributions.

Lemma 3.4. Let f be a locally integrable function such that (3.4) holds in the sense of distributions. Then

$$f = \Phi e^{\omega}$$

where Φ is an entire function and ω is a bounded, continuous function.

Formal Proof. Let

$$\omega(z) = -\frac{1}{\pi}\int d^2\zeta \frac{1}{z-\zeta} B(\zeta)\beta(\zeta)$$

where

$$\beta(\zeta) = \begin{cases} \overline{f(\zeta)}/f(\zeta), & f(\zeta) \neq 0 \\ 1, & f(\zeta) = 0 \end{cases}$$

An elementary convolution estimate shows that ω is Hölder-continuous. Clearly, ω is analytic outside the support of B and decays like z^{-1}, so the boundedness of ω follows.

Multiplying the differential equation by $e^{-\omega}$ and using the fact $\partial_{\bar{z}}\omega = -B\beta$, we have $\partial_{\bar{z}}(fe^{-\omega}) = 0$, so $fe^{-\omega}$ must be an entire function. This completes the "proof."

Remark. The proof is only formal because the elementary rules

$$\partial_{\bar{z}}e^{-\omega} = -e^{-\omega}\,\partial_{\bar{z}}\omega, \qquad \partial_{\bar{z}}(fe^{-\omega}) = f\,\partial_{\bar{z}}e^{-\omega} + e^{-\omega}\,\partial_{\bar{z}}f$$

have to be verified. Since ω is not absolutely continuous, these rules are not obvious, but Vekua proves them. (See the proof of the Basic Lemma on p. 144 of Ref. 25 and the theorems mentioned therein.)

Following Vekua, we introduce a function space that plays a central role in the theory. For ν real and $p \geqslant 1$ we define $L_{p,\nu}$ as the space of functions f such that

$$\|f\|_{p,\nu} \equiv \left[\int_{|z|\leq 1} |f(z)|^p\, d^2z\right]^{1/p} + \left[\int_{|z|\leq 1} |z|^{-\nu p}|f(z^{-1})|^p\, d^2z\right]^{1/p} < \infty$$

Note that B lies in every such space and that $L_{p,\nu} \subset L^p_{\text{loc}}$. These spaces were introduced because Vekua wished to isolate the local regularity condition and the decay condition at infinity that must be imposed on the variable coefficients of such equations as

$$\partial_{\bar{z}} f + Af + B\bar{f} = g$$

in order to obtain a reasonable existence theory for solutions. He assumes that the coefficients are in $L_{p,2}$, $p > 2$. The $L_{p,v}$ spaces are also appropriate for the integral operator P in our study of (3.3), as the following lemma makes clear. (In the next two lemmas B replaces ih in our definition of P.)

Lemma 3.5. P is a compact operator on the Banach space $L_{q,0}$ for arbitrary $q > 2$. Moreover, for $0 < \alpha < 1$ and $f \in L_{q,0}$, Pf is Hölder-continuous with exponent α and decays like z^{-1} at infinity.

This result follows from Theorem 1.25 of Ref. 25 together with the fact that $B \in L_{p,2}$ for arbitrary $p > 2$. The latter property of Pf is an obvious consequence of the compactness of supp B; also, Pf is actually analytic outside supp B.

Lemma 3.6. For $q > 2$ and $g \in L_{q,0}$ there exists a unique $f \in L_{q,0}$ satisfying Eq. (3.3).

Proof. Our reasoning is similar to that given on p. 156 of Ref. 25. By the Fredholm alternative, it suffices to show that the homogeneous equation $f + Pf = 0$ has only the zero solution in $L_{q,0}$. By Lemma 3.5, f is Hölder-continuous and decays at infinity like z^{-1}. Clearly, f also satisfies (3.4) in the sense of distributions. By Lemma 3.4, we know that $f = \Phi e^{\omega}$, where Φ is an entire function and ω is a bounded, continuous function. Hence $\Phi \equiv 0$. ■

Our third step is to prove the first theorem, and in doing so we will alternate freely between complex notation and real vector notation.

Proof of Theorem 3.1. In order to see that $S_0 h$ is a $\mathscr{C}_4$ operator on $\mathscr{H}_0$, note that

$$(S_0 h)^* = D^{-1}(S_0 h)^\dagger D$$

where $D = |p|$ in momentum space. Thus

$$(S_0 h)^* S_0 h = D^{-1} h S_0^\dagger D S_0 h = D^{-1} h D^{-1} h$$

Since h is a bounded function with compact support, it is clear that (with $x_5 = x_1$)

$$\begin{aligned} \operatorname{Tr}[(S_0 h)^* S_0 h]^2 &= 2 \int \prod_{i=1}^{4} d^2 x_i \prod_{i=1}^{4} h(x_i) \prod_{i=1}^{4} D^{-1}(x_i - x_{i-1}) \\ &\leqslant \text{const} \times \int_{(\operatorname{supp} h^4)} \prod_{i=1}^{4} d^2 x_i \prod_{i=1}^{4} D^{-1}(x_i - x_{i+1}) \end{aligned}$$

As

$$D^{-1}(x) = (1/2\pi)^2 \int d^2p\, e^{ix\cdot p}|p|^{-1}$$

it is easy to see by rotation invariance and scaling that

$$D^{-1}(x) = \text{const}/|x|$$

so the above integral must be finite.

Having shown that S_0h is compact on $\mathscr{H}_0$, we may now apply the Fredholm alternative: in order to show that $1 - S_0h$ is invertible, it suffices to show that $1 - S_0h$ has dense range. To this end let v be a vector whose components are real-valued C_0^∞ functions. Clearly it is enough to find a $u \in \mathscr{H}_0$ such that

$$(1 - S_0h)u = v$$

and we may impose the additional requirement that the components of u be real-valued. Thus in complex notation this equation reads

$$u + Pu = v$$

where $u = (u_0, u_1)$ and $v = (v_0, v_1)$ are identified with $u_0 + iu_1$ and $v_0 + iv_1$, respectively. By Lemma 3.6 there exists a unique such u in $L_{q,0}$ for arbitrary $q > 2$. We need only to show that u (as a vector) lies in $\mathscr{H}_0$. Since v (as a vector) certainly lies in $\mathscr{H}_0$, it will be sufficient to show that $S_0hu \in \mathscr{H}_0$.

Since h has compact support, $hu \in L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, so $\widehat{hu}$ lies in this space also. Hence

$$\begin{aligned}
\int d^2p\, |p|\, |\widehat{S_0hu}(p)|^2 &= \int d^2p\, |p|^{-3}|p\widehat{hu}(p)|^2 \\
&= \int d^2p\, |p|^{-1}|\widehat{hu}(p)|^2 \\
&\leqslant \int_{|p|\geq 1} d^2p\, |\widehat{hu}(p)|^2 + \int_{|p|\leq 1} d^2p\, |p|^{-1}|\widehat{hu}(p)|^2 \\
&< \infty
\end{aligned}$$

(The second integral is finite because $|p|^{-1}$ has an integrable singularity and hu is smooth.) ■

Our next step is to examine the properties of $S'(x, y)$. *Assuming* the existence of this kernel, we proceed formally down to Eq. (3.6a). First note that

$$S'(x, y) - \int d^2\zeta\, S_0(x, \zeta)h(\zeta)S'(\zeta, y) = S_0(x, y)$$

If we set

$$S'(x, y) \equiv \begin{bmatrix} \text{Re } X_2(z, t) & -\text{Re } X_1(z, t) \\ \text{Im } X_2(z, t) & -\text{Im } X_1(z, t) \end{bmatrix} \tag{3.5}$$

where $z = x_0 + ix_1$, $t = y_0 + iy_1$, it is easy to check that

$$X_1(z, t) + \frac{i}{\pi}\int d^2\zeta \frac{h(\zeta)}{z - \zeta} \overline{X_1(\zeta, t)} = \frac{1}{\pi(z - t)}$$

and

$$X_2(z, t) + \frac{i}{\pi}\int d^2\zeta \frac{h(\zeta)}{z - \zeta} \overline{X_2(\zeta, t)} = \frac{-i}{\pi(z - t)}$$

We wish to examine arbitrary, real, linear combinations $X = c_1X_1 + c_2X_2$ of X_1, X_2. Clearly

$$X(z, t) + \frac{i}{\pi}\int d^2\zeta \frac{h(\zeta)}{z - \zeta} \overline{X(\zeta, t)} = \frac{c}{\pi(z - t)} \tag{3.6a}$$

where $c = c_1 - ic_2$. What we will do logically is establish existence and uniqueness of $X(z, t)$ in a certain sense and then show that the matrix in (3.5) is indeed the kernel of $S' = (1 - S_0h)^{-1}S_0$. Theorems 3.2 and 3.3 will follow from the properties of $X(z, t)$.

Let t be fixed as we examine Eq. (3.6a). Set

$$W(z, t) = \pi c^{-1}(z - t)X(z, t)$$

Then the equation we wish to solve becomes

$$W(z, t) + \frac{1}{\pi}(z - t)\int d^2\zeta \frac{B(\zeta)}{(z - \zeta)(\zeta - t)} \overline{W(\zeta, t)} = 1$$

where

$$B(\zeta) = ih(\zeta)\bar{c}(\zeta - t)/c(\bar{\zeta} - \bar{t})$$

Clearly, the equation can be rewritten as

$$W(z, t) + \frac{1}{\pi}\int d^2\zeta \frac{B(\zeta)}{\zeta - t} \overline{W(\zeta, t)} + \frac{1}{\pi}\int d^2\zeta \frac{B(\zeta)}{z - \zeta} \overline{W(\zeta, t)} = 1 \tag{3.6b}$$

Since the constant function 1 lies in $L_{q,0}$, it follows from a modification of Lemma 3.6[3] that there is a unique solution of (3.6b) in $L_{q,0}$. This solution W also satisfies the differential equation

$$\partial_{\bar{z}}W(z, t) + B(z)\overline{W(z, t)} = 0$$

[3] The proof of Lemma 3.6 is not significantly altered. The perturbation of Eq. (3.3) is just a one-dimensional operator on $L_{q,0}$, so the operator in Eq. (3.6b) is still compact. When one considers the corresponding homogeneous equation, the entire function that arises is still bounded and vanishes at $z = t$ instead of vanishing at infinity.

and so by Lemma 3.4,

$$W(z, t) = \Phi(z, t)e^{\omega(z,t)}$$

where $\Phi(\cdot, t)$ is entire and $\omega(\cdot, t)$ is bounded and continuous. It follows from Eq. (3.6b) and Lemma 3.5 that $W(\cdot, t)$ is bounded, so by Liouville's theorem $\Phi(z, t)$ depends only on t. If we absorb this function of t into the exponential, we have $\omega(t, t) = 0$, since $W(t, t) = 1$. We have proven the following:

Lemma 3.7. Equation (3.6a) has a unique solution $X(z, t)$ with the property that $(z - t)X(z, t)$ lies in $L_{q,0}$. This solution has the representation

$$X(z, t) = \frac{c}{\pi(z - t)} e^{\omega(z,t)} \tag{3.7}$$

where $\omega(\cdot, t)$ is a bounded, continuous function and $\omega(t, t) = 0$.

The argument we have just given is very similar to the reasoning involved in the proof of Theorem 3.13 in Ref. 25.

Let X_1 and X_2 be the solutions for $c = 1$ and $c = -i$, respectively. Following Vekua, we refer to this pair of functions as the *system of fundamental generalized analytic functions for the equation*

$$\partial_{\bar{z}} f + ih\bar{f} = 0 \tag{3.8}$$

with pole at t. This pair of functions has an elementary property that will be very important to us:

Lemma 3.8. $\operatorname{Im} \overline{X_1(z, t)} X_2(z, t) < 0$.

Proof. Let c_1, c_2 be real numbers and $X(z, t)$ be the solution for $c = c_1 - ic_2$. Then, inspecting (3.5), we see that, by uniqueness,

$$X = c_1 X_1 + c_2 X_2$$

It follows from (3.7) that $c_1X_1 + c_2X_2$ does not vanish anywhere unless $c_1 = c_2 = 0$. Thus X_1, X_2 are pointwise, linearly independent over the real numbers. Since $\operatorname{Im} \bar{X}_1 X_2$ is just the pointwise cross product of X_1, X_2 considered as real vectors, it follows that $\operatorname{Im} \bar{X}_1 X_2$ does not vanish anywhere. As this real quantity is continuous for $z \neq t$, it suffices to show that it is negative somewhere.

We have the representations

$$X_1(z, t) = \frac{1}{\pi(z - t)} e^{\omega_1(z,t)} \tag{3.9a}$$

$$X_2(z, t) = \frac{-i}{\pi(z - t)} e^{\omega_2(z,t)} \tag{3.9b}$$

so

$$\operatorname{Im} \overline{X_1(z,t)} X_2(z,t) = -\frac{1}{\pi^2 |z-t|^2} \operatorname{Re} \exp[\overline{\omega_1(z,t)} + \omega_2(z,t)]$$

Since $\omega_1(t,t) = \omega_2(t,t) = 0$, it follows that $\lim_{z \to t} \operatorname{Im} \bar{X}_1 X_2 = -\infty$. ■

So far we do not know how regular $X_1(z,t)$ and $X_2(z,t)$ are with respect to t. Vekua resolves this question for the equation

$$\partial_{\bar{z}} f + Af + B\bar{f} = 0 \tag{3.10}$$

by considering the adjoint equation

$$\partial_{\bar{z}} f - Af - \bar{B}\bar{f} = 0 \tag{3.10$'$}$$

Let X_1', X_2' be the system of fundamental generalized analytic functions for the adjoint equation (3.10′). Then, using the (generalized) Cauchy integral formula, Vekua shows that (see Ref. 25, pp. 172–177)

$$\begin{aligned} X_1(z,t) &= -\operatorname{Re} X_1'(t,z) - i \operatorname{Re} X_2'(t,z) \\ X_2(z,t) &= -\operatorname{Im} X_1'(t,z) - i \operatorname{Im} X_2'(t,z) \end{aligned} \tag{3.11}$$

In particular, $X_k(z,t)$ is continuous in t on $\mathbb{C}\backslash\{z\}$.

Now notice that in our case $A = 0$, $B = ih$ with h real. Hence (3.10) and (3.10′) are identical and the relations (3.11) become the symmetries

$$\begin{aligned} X_1(z,t) &= -\operatorname{Re} X_1(t,z) - i \operatorname{Re} X_2(t,z) \\ X_2(z,t) &= -\operatorname{Im} X_1(t,z) - i \operatorname{Im} X_2(t,z) \end{aligned} \tag{3.12}$$

We are now ready to prove Theorem 3.2; we do it by showing that the matrix

$$\tilde{S}(x,y) \equiv \begin{pmatrix} \operatorname{Re} X_2(z,t) & -\operatorname{Re} X_1(z,t) \\ \operatorname{Im} X_2(z,t) & -\operatorname{Im} X_1(z,t) \end{pmatrix}, \qquad z = x_0 + ix_1, \quad t = y_0 + iy_1$$

is the kernel of S' of (3.1).

Proof of Theorem 3.2. Let v be a vector whose components are real-valued C_0^∞ functions. It suffices to show that $S_0 v \in \mathscr{H}_0$ and

$$S'v(x) = \int \tilde{S}(x,y) v(y)\, d^2y \tag{3.13}$$

In complex notation the rhs reads

$$-\int d^2t\, X_1(z,t) \operatorname{Im} v(t) + \int d^2t\, X_2(z,t) \operatorname{Re} v(t)$$

which we will denote by $u(z)$. It follows from the integral equation (3.6a) and the definition of X_1, X_2 that

$$u(z) + \frac{i}{\pi}\int d^2\zeta \frac{h(\zeta)}{z - \zeta} \overline{u(\zeta)} = -\frac{i}{\pi}\int d^2t \frac{\overline{v(t)}}{z - t} \tag{3.14}$$

This is the basic integral equation (3.3), where

$$g(z) = -\frac{i}{\pi}\int d^2t \frac{\overline{v(t)}}{z - t}$$

Since v has compact support, g is certainly in $L_{q,0}$, so (3.14) has a unique solution in $L_{q,0}$, and it is clear that u is that solution. In vector notation (3.14) reads [see Eq. (3.1)]:

$$u - S_0 hu = S_0 v$$

The proof that $S_0 v$ and $S_0 hu$ are in the space $\mathscr{H}_0$ is identical to the argument in the proof of Theorem 3.1. Thus $u \in \mathscr{H}_0$ and so

$$u = (1 - S_0 h)^{-1} S_0 v$$

which is exactly Eq. (3.13). ■

Our strategy for proving Theorem 3.2 has given us a hold on the kernel of S' and we are now able to prove the last theorem.

Proof of Theorem 3.3.

$$\begin{aligned} \operatorname{tr} S'(x, y)S'(y, x) = {} & \operatorname{Re} X_2(z, t) \operatorname{Re} X_2(t, z) - \operatorname{Re} X_1(z, t) \operatorname{Im} X_2(t, z) \\ & - \operatorname{Im} X_2(z, t) \operatorname{Re} X_1(t, z) + \operatorname{Im} X_1(z, t) \operatorname{Im} X_1(t, z) \end{aligned}$$

It follows from Eqs. (3.12) that

$$\begin{aligned} \operatorname{tr} S'(x, y)S'(y, x) &= -\operatorname{Im} X_1(z, t)\overline{X_2(z, t)} + \operatorname{Im} \overline{X_1(z, t)} X_2(z, t) \\ &= 2 \operatorname{Im} \overline{X_1(z, t)} X_2(z, t) \end{aligned}$$

We conclude from Lemma 3.8 that $\operatorname{tr} S'(x, y)S'(y, x) < 0$. ■

4. LATTICE APPROXIMATION FOR Y_2.

Throughout this section we adopt the convention $m_f = m_b = m$.

We wish to prove the convergence of the boson lattice approximation for the Yukawa$_2$ model, where the approximation is introduced in the usual way (Ref. 5, §IV): $\delta > 0$ is the distance between lattice points; with each lattice point $n\delta \in (\delta Z)^2$ we associate the random variable $q_n \equiv \phi(f_{\delta,n})$, where

$$f_{\delta,n}(x) = \frac{1}{(2\pi)^2}\int_{T_\delta} e^{ik\cdot(x - n\delta)} \mu(k)\mu_\delta(k)^{-1}\, d^2k$$

with

$$\mu_\delta(k)^2 = m^2 + \delta^{-2}(4 - 2\cos k_0\delta - 2\cos k_1\delta) \tag{4.1}$$

and T_δ the square $T_\delta = [-\pi/\delta, \pi/\delta]^2$. If $x \in \mathbb{R}^2$ and $n\delta$ is the lattice point closest to x (with a suitable convention if x lies midway between lattice points), we write $[x] = n$. The lattice cutoff boson field is then defined to be

$$\phi_\delta(x) \equiv q_{[x]} \tag{4.2a}$$

The associated covariance

$$C_\delta(x, y) \equiv \int \phi_\delta(x)\phi_\delta(y)\, d\mu = \frac{1}{(2\pi)^2}\int_{T_\delta} e^{ik\delta\cdot([x]-[y])}\mu_\delta(k)^{-2}\, dk \tag{4.2b}$$

can be identified[5] with $(-\Delta_\delta + m^2)^{-1}$, where Δ_δ is the finite-difference Laplacian on $(\delta Z)^2$.

We impose no lattice cutoff on the Fermi propagator, i.e.,

$$K_\delta(x, y) = S(x, y)\phi_\delta(y)\chi_\Lambda(y)$$

and the cutoff density is [see (1.9)]

$$\rho_\delta = \rho(K_\delta) = \det_3(1 - K_\delta)e^{-B_\delta}$$

where $B_\delta = \frac{1}{2}{:}\mathrm{Tr}(K_\delta{}^2 + K_\delta{}^\dagger K_\delta){:}$. Our main result is:

Theorem 4.1. For any $p < \infty$, $\rho_{\delta_n} \to \rho$ in $L^p(d\mu)$ for some sequence $\{\delta_n\}$ converging to 0.

Remark. MacDermot[13] has established a lattice approximation differing from ours in that S is also replaced by a finite-difference Green's function $(-\beta\cdot\partial_{\delta_f} + m)^{-1}$, where $\delta_f = 2\delta$. In our case it is essential that $\delta_f = 0$, so that we cannot use MacDermot's approximation. We believe that the techniques of this section permit a lattice approximation like MacDermot's with $\delta_f > 0$ and with no connection assumed between δ_f and δ.

By the strategy outlined in Section 2 (see Lemma 2.4), Theorem 4.1 follows from the following three lemmas.

Lemma 4.2. For any $p < \infty$, $\rho_\delta \in L^p(d\mu)$ with bounds uniform in δ.

Lemma 4.3. $K_\delta \to K$ in $\mathscr{C}_{3,3}$ as $\delta \to 0$.

Lemma 4.4. $B_\delta \to B$ in $L^2(d\mu)$ as $\delta \to 0$.

Since we shall apply Theorem 2.1 to the major task of proving Lemma 4.2, most of the effort in this section is directed toward verifying the hypotheses of Theorem 2.1 for a convenient ultraviolet cutoff. Let

$$q(k) = \frac{\delta^2}{2\pi}\sum_n e^{ik\cdot n\delta}q_n, \qquad k \in T_\delta$$

so that

$$q_n = (1/2\pi)\int_{T_\delta} e^{-ik\cdot n\delta}\tilde{q}(k)\, dk \tag{4.3}$$

and

$$\int \overline{\tilde{q}(k)}q(k')\, d\mu = \delta(k - k')\mu_\delta(k)^{-2}, \qquad k, k' \in T_\delta \tag{4.4}$$

We define our ultraviolet cutoff by modifying the integral

$$\phi_\delta(x) \equiv q_{[x]} = (1/2\pi)\int_{T_\delta} e^{-ik\cdot[x]\delta}\tilde{q}(k)\,dk$$

in the following way: set

$$\phi_{\delta,j}(x) = (1/2\pi)\int_{T_\delta} e^{-ik\cdot[x]\delta}\tilde{q}(k)e^{-\mu_\delta(k)/2\kappa}\,dk \tag{4.5a}$$

$$\delta\phi_j(x) = \phi_\delta(x) - \phi_{\delta,j}(x) \tag{4.5b}$$

where $\kappa = e^j$. The corresponding covariance operator is

$$C_{\delta,j}(x, y) \equiv \int \phi_{\delta,j}(x)\phi_{\delta,j}(y)\,d\mu = \frac{1}{(2\pi)^2}\int_{T_\delta} e^{ik\delta\cdot([x]-[y])}\mu_\delta(k)^{-2}e^{-\mu_\delta(k)/\kappa}\,dk \tag{4.5c}$$

Notice that $C_{\delta,j}$ satisfies the monotonicity relation (2.6b) but not the relation (2.6a). This deficiency entails only a minor modification of the proof of Theorem 2.1 and so we shall continue to appeal to Theorem 2.1 as if (2.6a) held. But why have we introduced the j cutoff as in (4.5a) instead of as a sharp cutoff (i.e., $|k| \leqslant \kappa$) for which (2.6a) obviously holds? The reason is that we do not want the ultraviolet cutoff to interfere with the known small-distance bound on $C_\delta(x, y)$; see part (c) of the following lemma, which lists useful properties of $C_{\delta,j}$:

Lemma 4.5. (a) If $i \leqslant j$, then $C_{\delta,i} \leqslant C_{\delta,j}$ as operators on L^2.
(b) There is a constant c independent of δ and j such that

$$|C_{\delta,j}(x, y)| \leqslant c\min(j, \log\delta^{-1}) \tag{4.6}$$

(c) There is a constant c independent of δ and j such that

$$|C_\delta(x, y)| + |C_{\delta,j}(x, y)| \leqslant c\log(2 + |x - y|^{-1}) \tag{4.7}$$

Proof. (a) is obvious. (b) follows from the bound (Ref. 5, Lemma IV.2)

$$(\pi/2)\mu_\delta(k) \geqslant \mu(k) \equiv (k^2 + m^2)^{1/2}, \qquad k \in T_\delta \tag{4.8}$$

For by (4.8) (c denotes various positive constants)

$$\begin{aligned} |C_{\delta,j}| &\leqslant c\int_{\mathbb{R}^2} \mu(k)^{-2}e^{-c\mu(k)/\kappa}\,dk \\ &= 2\pi c\int_m^\infty \mu^{-1}e^{-c\mu/\kappa}\,d\mu \\ &= 2\pi c\int_{m/\kappa}^\infty e^{-cx}\frac{dx}{x} = O(\log\kappa) = O(j) \end{aligned}$$

Alternatively, we have

$$|C_{\delta,j}| \leqslant c\int_{T_\delta} \mu(k)^{-2}\, dk = O(\log \delta^{-1})$$

and this yields (4.6).

For part (c) we follow the proof of Lemma IX.8 of Ref. 32, where it is shown that

$$|C_\delta(x, y)| \leqslant c \log(2 + r^{-1}) \tag{4.9}$$

where $r = \delta \max([x_0] - [y_0], [x_1] - [y_1])$. The proof involves a contour-shifting argument which is unaltered by the presence of the extra factor $\exp[-\mu_\delta(k)/\kappa]$ since, for fixed k_0, $\mu_\delta(k)$ is analytic in k_1 in a strip about the real axis and is periodic in k_1 with period $2\pi/\delta$. Thus we obtain the bound (4.9) for $C_{\delta,j}$ uniformly in j and δ. This implies (4.7) if $r \neq 0$; if $r = 0$, then $|x - y| \leqslant \sqrt{2}\delta$ and (4.7) follows from (4.6). ■

We define the tails as in (2.7): $K_{\delta,j} = S\chi_\Lambda\phi_{\delta,j}$,

$$\delta K_j = K_\delta - K_{\delta,j} = S\chi_\Lambda\, \delta\phi_j \tag{4.10a}$$

$$\delta B_j = :\mathrm{Tr}(K_\delta + K_\delta^\dagger)\, \delta K_j: \tag{4.10b}$$

The various hypotheses of Theorem 2.1 are verified in Lemmas 4.6 and 4.9–4.11 below.

Lemma 4.6. For $\epsilon > 0$, the following estimates are uniform in $\delta > 0$:

$$\|D^{-\epsilon}K_\delta\|_{2,2} < \infty \tag{4.11a}$$

$$\|D^{-\epsilon}\, \delta K_j\|_{2,2} \leqslant ce^{-cj} \tag{4.11b}$$

Remark. For notational convenience only, we take Λ to be a square centered at the origin and with side length $L = (2N + 1)\delta$ for a positive integer N. Moreover, since we are taking $m_b = m_f$, we shall often write $\mu(p)$ instead of $D(p)$.

Proof. (4.11a) will follow from a trivial modification of our proof of (4.11b), so that we restrict ourselves to the latter bound. We compute that

$$\begin{aligned}\|D^{-\epsilon}\delta K_j\|_2^2 &= \mathrm{Tr}\, D^{-1}\delta\phi_j\chi_\Lambda D^{-1-2\epsilon}\delta\phi_j\chi_\Lambda \\ &= \frac{2}{(2\pi)^4}\int_{\Lambda^2} dx\, dy \int_{\mathbb{R}^4} dp\, dp' \int_{T_\delta^2} dk\, dk' \\ &\quad \times \exp\{ip\cdot(x - y) + ip'\cdot(y - x) + ik\cdot[x]\delta - ik'\cdot[y]\delta\} \\ &\quad \times D(p)^{-1}D(p')^{-1-2\epsilon}t_j(k)t_j(k')\overline{\tilde{q}(k)}\tilde{q}(k')\end{aligned}$$

where

$$t_j(k) = 1 - e^{-\mu_\delta(k)/2\kappa}$$

If Δ_0 denotes the lattice square centered at 0 and with area δ^2, then

$$\frac{1}{2\pi}\int_\Lambda dx \exp(ipx - ip'x + ik[x]\delta)$$
$$= \frac{\delta^2}{2\pi}\sum_{n\delta\in\Lambda} \exp[i(p - p' + k)n\delta]\frac{1}{\delta^2}\int_{\Delta_0} dx \exp[i(p - p')x]$$
$$= \tilde{\chi}(p - p' + k)\psi(p - p')$$

where

$$\tilde{\chi}(p) = \frac{\delta^2}{2\pi}\sum_{n\delta\in\Lambda} e^{ip\cdot n\delta} = \frac{\delta^2}{2\pi}\prod_{i=0}^{1}\frac{\sin[p_i\delta(N + \frac{1}{2})]}{\sin(p_i\delta/2)}$$

and

$$\psi(p) = \frac{1}{\delta^2}\int_{\Delta_0} dx\, e^{ipx} = \prod_{i=0}^{1}\frac{\sin(p_i\delta/2)}{p_i\delta/2}$$

With this notation we have from (4.4)

$$\|D^{-\epsilon}\delta K_j\|_{2,2}^2 = \int \|D^{-\epsilon}\delta K_j\|_2{}^2\, d\mu$$
$$= \frac{2}{(2\pi)^2}\int dp\, dp'\int_{T_\delta} dk\, \mu_\delta(k)^{-2}t_j(k)^2D(p)^{-1}D(p')^{-1-2\epsilon}$$
$$\times\, \psi(p - p')^2\tilde{\chi}(p - p' + k)^2 \tag{4.12}$$

Since $1 - e^{-u} \leqslant u^\alpha$ for $u \geqslant 0$ and $0 \leqslant \alpha \leqslant 1$,

$$0 \leqslant t_j(k) \leqslant \mu_\delta{}^\alpha/(2\kappa)^\alpha \tag{4.13}$$

where we choose $\alpha < \epsilon$. Making the change of variable $p \to p + p'$ and using the estimate (see Appendix C, Lemma C5)

$$\int d^2p'\, D(p + p')^{-1}D(p')^{-1-2\epsilon} \leqslant \text{const } D(p)^{-2\epsilon}$$

and the estimate (4.9), we obtain

$$\|D^{-\epsilon}\delta K_j\|_{2,2}^2 \leqslant c\kappa^{-2\alpha}\int_{T_\delta} dk\, \mu(k)^{-2+2\alpha}\int dp\, D(p)^{-2\epsilon}\psi(p)^2\tilde{\chi}(p + k)^2 \tag{4.14}$$

To show that (4.14) is finite, uniformly in δ, it suffices to show that ($\mu = D$)

$$\int dp\, \mu(p)^{-2\epsilon}\psi(p)^2\tilde{\chi}(p + k)^2 \leqslant c\mu(k)^{-2\epsilon} \tag{4.15}$$

We wish to estimate the left side of (4.15) in terms of one-dimensional integrals. It is a reasonable abuse of notation to use the letters μ, ψ, and $\tilde{\chi}$ to denote the respective one-dimensional analogs: $\mu(k_i)^2 \equiv m^2 + k_i^2$,

$$\tilde{\chi}(k_i) \equiv \frac{\delta}{(2\pi)^{1/2}} \sum_{n=-N}^{N} e^{ik_i n\delta} \tag{4.16a}$$

$$= \frac{\delta}{(2\pi)^{1/2}} \frac{\sin[k_i\delta(N+\frac{1}{2})]}{\sin(k_i\delta/2)} \tag{4.16b}$$

$$\psi(k_i) \equiv \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} e^{ik_ix}\,dx = \frac{\sin(k_i\delta/2)}{k_i\delta/2} \tag{4.16c}$$

Now it follows from the positivity of perfect squares that $\mu(k)^2 \geqslant \mu(k_0)\mu(k_1)$; moreover, $\psi(k) = \psi(k_0)\psi(k_1)$ and $\tilde{\chi}(k) = \tilde{\chi}(k_0)\tilde{\chi}(k_1)$. Therefore, we need only consider

$$\int dp\, \mu(p)^{-\epsilon}\psi(p)^2\tilde{\chi}(p+k)^2, \qquad |k| \leqslant \pi\delta^{-1} \tag{4.17}$$

where p and k now denote one-dimensional variables. This expression can be rewritten as

$$\sum_n \int_{\pi/\delta}^{-\pi/\delta} dp\, \mu(p-2\pi n\delta^{-1})^{-\epsilon}\psi(p-2\pi n\delta^{-1})^2\tilde{\chi}(p+k)^2$$

because $\tilde{\chi}$ is periodic. Note that for $n \neq 0$ and $|p| \leqslant \pi\delta^{-1}$ we have

$$\mu(p-2\pi n\delta^{-1})^{-\epsilon} \leqslant \mu(\pi\delta^{-1})^{-\epsilon} \leqslant \mu(k)^{-\epsilon}, \qquad |k| \leqslant \pi\delta^{-1}$$

and

$$|\psi(p-2\pi n\delta^{-1})| = \frac{|\sin(p\delta/2-\pi n)|}{|p\delta/2-\pi n|} \leqslant \frac{c}{|n|}$$

Hence (4.17) is dominated by

$$\int_{-\pi/\delta}^{\pi/\delta} dp\, \mu(p)^{-\epsilon}\psi(p)^2\tilde{\chi}(p+k)^2 + \text{const}\, \mu(k)^{-\epsilon}\left(\sum_{n\neq 0}\frac{1}{n^2}\right)\int_{-\pi/\delta}^{\pi/\delta} dp\, \tilde{\chi}(p+k)^2$$

for $|k| \leqslant \pi\delta^{-1}$. It follows from (4.16a) that

$$\int_{-\pi/\delta}^{\pi/\delta} dp\, \tilde{\chi}(p+k)^2 = \delta(2N+1) = L \tag{4.18}$$

which is independent of δ, so it remains only to estimate the first term, which we break up into two integrals:

$$\int_{\pi/2\delta \leq |p| \leq \pi/\delta} dp\, \mu(p)^{-\epsilon}\psi(p)^2\tilde{\chi}(p+k)^2 + \int_{-\pi/2\delta}^{\pi/2\delta} dp\, \mu(p)^{-\epsilon}\psi(p)^2\tilde{\chi}(p+k)^2 \tag{4.19}$$

Now, for $|p| \geqslant \pi/2\delta$,

$$\mu(p)^{-\epsilon} \leqslant \mu(\tfrac{1}{2}\pi\delta^{-1})^{-\epsilon} \leqslant \text{const}\, \mu(\pi\delta^{-1})^{-\epsilon} \leqslant \text{const}\, \mu(k)^{-\epsilon}$$

for $|k| \leqslant \pi\delta^{-1}$. Since $|\psi(p)| \leqslant 1$, it follows that the first integral is dominated by

$$\begin{aligned} &\text{const}\, \mu(k)^{-\epsilon} \int_{\pi/2\delta \leq |p| \leq \pi/\delta} dp\, \tilde{\chi}(p + k)^2 \\ &\leqslant \text{const}\, \mu(k)^{-\epsilon} \int_{-\pi/\delta}^{\pi/\delta} dp\, \tilde{\chi}(p + k)^2 \\ &= \text{const}\, \mu(k)^{-\epsilon} L \end{aligned}$$

In order to estimate the second term of (4.19), we pause for the following lemma:

Lemma 4.7. If $|u| \leqslant \frac{3}{2}\pi\delta^{-1}$, then $|\tilde{\chi}(u)| \leqslant \text{const}\,(1 + |u|)^{-1}$, where the constant is independent of δ.

Proof. If $|u| \leqslant \frac{3}{2}\pi\delta^{-1}$, then $|u\delta/2| \leqslant \frac{3}{4}\pi$, so by (4.16b),

$$|\tilde{\chi}(u)| = \frac{\delta}{(2\pi)^{1/2}} \frac{|\sin[u\delta(N + \frac{1}{2})]|}{|\sin(u\delta/2)|} \leqslant \frac{\delta}{(2\pi)^{1/2}} \frac{1}{\text{const}\, |\delta u|} = \frac{\text{const}}{|u|}$$

because $(\sin y)/y$ is bounded away from zero for $|y| \leqslant 3\pi/4$. On the other hand, by (4.16a),

$$|\tilde{\chi}(u)| \leqslant \frac{\delta}{(2\pi)^{1/2}} (2N + 1) = \frac{L}{(2\pi)^{1/2}} \quad \blacksquare$$

Proof of Lemma 4.6 (concluded). In our situation $|k| \leqslant \pi\delta^{-1}$, so if $|p| \leqslant \frac{1}{2}\pi\delta^{-1}$, it follows from Lemma 4.7 that

$$|\tilde{\chi}(p + k)| \leqslant \text{const}\,(1 + |p + k|)^{-1}$$

Since $|\psi(p)| \leqslant 1$, we see that the second integral in (4.19) is dominated by

$$\begin{aligned} &\text{const} \int_{-\pi/2\delta}^{\pi/2\delta} dp\, \mu(p)^{-\epsilon}(1 + |p + k|)^{-2} \\ &\leqslant \text{const} \int_{-\infty}^{\infty} dp\, \mu(p)^{-\epsilon}(1 + |p + k|)^{-2} \end{aligned}$$

By Lemma C1, the last integral is dominated by const $\mu(k)^{-\epsilon}$. In summary, we have shown that (4.17) has this decay in k, for $|k| \leqslant \pi\delta^{-1}$, with constants independent of δ. This establishes (4.15) and the lemma. $\blacksquare$

The next lemma to be verified is:

Lemma 4.8. For $\epsilon < \frac{1}{4}$ the following estimates are uniform in δ:

$$\|D^{\epsilon}K_{\delta}\|_{4,4} < \infty \tag{4.20a}$$

$$\|D^{\epsilon}\delta K_{j}\|_{4,4} \leqslant \text{const}\, e^{-cj} \tag{4.20b}$$

Remark. It is worth pointing out that the bounds (4.11a) and (4.20a) are transparent in configuration space if one uses the bound (4.7). For example,

$$\begin{aligned}\int \|D^{-\epsilon}K_{\delta}\|_{2}^{2}\, d\mu &= 2\int_{\Lambda} d^{2}x \int_{\Lambda} d^{2}y\, D^{-1}(x-y)D^{-1-2\epsilon}(x-y)C_{\delta}(x,y)\\ &\leqslant \text{const}\int_{\Lambda} d^{2}x\int_{\Lambda} d^{2}y\, D^{-1}(x-y)D^{-1-2\epsilon}(x-y)\log(|x-y|^{-1}+2)\end{aligned}$$

which is finite since $D^{-\alpha}(x) \sim |x|^{\alpha-2}$ at $x = 0$. The estimation of $\|D^{\epsilon}K_{\delta}\|_{4,4}$ is similar:

$$\begin{aligned}\int \|D^{\epsilon}K_{\delta}\|_{4}^{4}\, d\mu &= 2\int_{\Lambda^{4}} \prod_{i=1}^{4} d^{2}x_{i}\, D^{-1}(x_{1}-x_{2})D^{-1+2\epsilon}(x_{2}-x_{3})D^{-1}(x_{3}-x_{4})\\ &\quad\times D^{-1+2\epsilon}(x_{4}-x_{1})\{C_{\delta}(x_{1},x_{2})C_{\delta}(x_{3},x_{4}) + C_{\delta}(x_{1},x_{3})C_{\delta}(x_{2},x_{4})\\ &\quad + C_{\delta}(x_{1},x_{4})C_{\delta}(x_{2},x_{3})\}\end{aligned} \tag{4.21}$$

which is finite uniformly in δ, provided that $\epsilon < 1/2$.

It seems extremely difficult to prove (4.20) by direct computation in momentum space. Instead we shall rely on configuration-space computations such as those above to obtain bounds on quantities like $\|D^{\epsilon}K_{\delta,j}\|_{4,4}$ that are uniform in both δ and j; in this way we shall reduce the tail bound (4.20b) to a simpler $\mathscr{C}_{2,2}$ bound as in (4.11b).

Proof of Lemma 4.8. Once again, the proof of the first estimate is almost identical to the proof of the second. Consider

$$\begin{aligned}\|D^{\epsilon}\delta K_{j}\|_{4}^{4} &= \mathrm{Tr}(\delta K_{j}^{*}D^{2\epsilon}\delta K_{j}\delta K_{j}^{*}D^{2\epsilon}\delta K_{j})\\ &= \mathrm{Tr}(D^{\alpha}\delta K_{j}^{*}D^{2\epsilon}\delta K_{j}D^{\alpha}D^{-\alpha}\delta K_{j}^{*}D^{2\epsilon}\delta K_{j}D^{-\alpha})\end{aligned}$$

where $\alpha > 0$ is to be chosen later. By the Schwarz inequality,

$$\begin{aligned}\|D^{\epsilon}\delta K_{j}\|_{4}^{4} &\leqslant \|D^{\alpha}\delta K_{j}^{*}D^{2\epsilon}\delta K_{j}D^{\alpha}\|_{2}\|D^{-\alpha}\delta K_{j}^{*}D^{2\epsilon}\delta K_{j}D^{-\alpha}\|_{2}\\ &= \|D^{\epsilon}\delta K_{j}D^{\alpha}\|_{4}^{2}\|D^{\epsilon}\delta K_{j}D^{-\alpha}\|_{4}^{2}\\ &\leqslant \|D^{\epsilon}\delta K_{j}D^{\alpha}\|_{4}^{2}\|D^{\epsilon}\delta K_{j}D^{-\alpha}\|_{2}^{2}\end{aligned}$$

Hence

$$\|D^{\epsilon}\delta K_j\|_{4,4}^4 \leqslant \|D^{\epsilon}\delta K_j D^{\alpha}\|_{4,4}^2 \left(\int \|D^{\epsilon}\delta K_j D^{-\alpha}\|_2^4\, d\mu\right)^{1/2}$$
$$\leqslant \text{const}\, \|D^{\epsilon}\delta K_j D^{\alpha}\|_{4,4}^2 \|D^{\epsilon}\delta K_j D^{-\alpha}\|_{2,2}^2 \qquad (4.22)$$

by the Schwarz inequality and hypercontractivity. Now the computation of $\|D^{\epsilon}\delta K_j D^{-\alpha}\|_{2,2}^2$ is identical to the computation of $\|D^{-\epsilon}\delta K_j\|_{2,2}^2$ in (4.12) except that we have the replacements $D(p)^{-1} \to D(p)^{-1-2\alpha}$ and $D(p')^{-1-2\epsilon} \to D(p')^{-1+2\epsilon}$. Making the change of variable $p \to p + p'$ as before, we integrate out p' via the estimate

$$\int d^2p'\, D(p+p')^{-1-2\alpha} D(p')^{-1+2\epsilon} \leqslant \text{const}\, D(p)^{2\epsilon-2\alpha}$$

which holds if $\epsilon < \alpha < 1/2$ (see Lemma C5). The estimation now proceeds as before, with $\alpha - \epsilon$ replacing ϵ. Thus, we may conclude that

$$\|D^{\epsilon}\delta K_j D^{-\alpha}\|_{2,2} \leqslant \text{const}\, e^{-cj}$$

for some $c > 0$, where the constants are independent of δ (provided that $\epsilon < \alpha < 1/2$). Therefore, it is clear from (4.22) that we need only obtain a uniform bound on $\|D^{\epsilon}\delta K_j D^{\alpha}\|_{4,4}$. Clearly,

$$\|D^{\epsilon}\delta K_j D^{\alpha}\|_{4,4} \leqslant \|D^{\epsilon} K_{\delta} D^{\alpha}\|_{4,4} + \|D^{\epsilon} K_{\delta,j} D^{\alpha}\|_{4,4}$$

We estimate each of these terms by a configuration-space calculation. We obtain an expression like (4.21) except that D^{-1} is replaced by $D^{-1+2\alpha}$ and, in the case of the second term, C_{δ} is replaced by $C_{\delta,j}$. By (4.7) and Lemma C6 the resulting integrals are finite if $\alpha + \epsilon < 1/2$. This establishes (4.20) for $\epsilon < 1/4$. ■

Lemma 4.9. The following estimates are uniform in δ:

$$\|B_{\delta}\|_{L^2} < \infty \qquad (4.23a)$$

$$\|\delta B_j\|_{L^2} \leqslant \text{const}\, e^{-cj} \qquad (4.23b)$$

for some $c > 0$.

Proof. As before, the proof of (4.23a) is essentially contained in the proof of (4.23b). By (2.26)

$$\delta B_j = c\int_{\Lambda\times\Lambda} dx\, dy\, b(x-y) : \phi_{\delta}(x)\delta\phi_j(y):$$

where

$$\hat{b}(p) = \frac{2}{(2\pi)^3}\int \left(\frac{m^2 - q\cdot(q+p)}{D(q)^2 D(q+p)^2} + \frac{1}{D(q)^2}\right)$$
$$= O[\log(2 + |p|)] = O[\log(2+|p_0|) + \log(2+|p_1|)] \qquad (4.24)$$

as is shown in Appendix A of Ref. 16. As in the computations leading to (4.12), we find in momentum space that

$$\delta B_j = c\int dp \int_{T_\delta^2} dk\, dk'\, \hat{b}(p)\tilde{\chi}(p-k')\tilde{\chi}(p-k)\psi(p)^2 t_j(k') {:}\overline{\tilde{q}(k)}\tilde{q}(k'){:}$$

It follows from (4.4) and the fact that $\hat{b}$, $\tilde{\chi}$, and ψ are even that

$$\|\delta B_j\|_{L^2}^2 = c\int_{T_\delta^2} dk\, dk'\, \mu_\delta(k)^{-2}\mu_\delta(k')^{-2}[t_j(k)t_j(k') + t_j(k')^2]$$
$$\times \left[\int dp\, \hat{b}(p)\psi(p)^2\tilde{\chi}(p-k)\tilde{\chi}(p-k')\right]^2$$

We apply the estimates (4.8) and (4.24) and the estimate (4.13) to extract the desired factor $\kappa^{-2\alpha}$ for some $\alpha > 0$, and we are left with bounding the one-dimensional integral

$$\int_{-\pi/\delta}^{\pi/\delta} dk \int_{-\pi/\delta}^{\pi/\delta} dk'\, [\mu(k)^{-1+\alpha/2}\mu(k')^{-1+\alpha/2} + \mu(k)^{-1}\mu(k')^{-1+\alpha}]G_\delta(k,k')^2 \tag{4.25}$$

where

$$G_\delta(k,k') \equiv \left[\int dp \log(2+|p|)\psi(p)^2|\tilde{\chi}(p-k)\tilde{\chi}(p-k')|\right]$$

Here we adopt the same abuse of notation as in the proof of Lemma 4.6 [see (4.16) for the definitions of ψ and $\tilde{\chi}$]. We shall bound (4.25) when $\alpha < 1/2$. We break up the p integration into intervals as before:

$$G_\delta(k,k') = \sum_n \int_{-\pi/\delta}^{\pi/\delta} dp\, \psi(p - 2\pi n\delta^{-1})^2$$
$$\times \log(2 + |p - 2\pi n\delta^{-1}|)|\tilde{\chi}(p-k)\tilde{\chi}(p-k')|$$

This time, we need the more subtle inequality: for $n \neq 0$ and $|p| \leqslant \pi/\delta$,

$$|\psi(p)| = \frac{|\sin(p\delta/2 - \pi n)|}{|p\delta/2 - \pi n|} = \left|\frac{\sin(p\delta/2)}{p\delta/2 - \pi n}\right| \leqslant \text{const}\,\frac{|p|^\lambda\delta^\lambda}{|n|}$$

where λ is to be chosen below, $0 < \lambda \leqslant 1$. Also, $|\psi(p)| \leqslant 1$, so we have

$$G_\delta(k,k') \leqslant \int_{-\pi/\delta}^{\pi/\delta} dp \log(2+|p|)|\tilde{\chi}(p-k)\tilde{\chi}(p-k')| + \text{const} \times \delta^{2\lambda}$$
$$\times \left(\sum_{n\neq 0} \frac{\log|n| + \log\delta^{-1}}{|n|^2}\right)\int_{-\pi/\delta}^{\pi/\delta} dp\, |p|^{2\lambda}|\tilde{\chi}(p-k)\tilde{\chi}(p-k')|$$
$$\leqslant c\int_{-\pi/\delta}^{\pi/\delta} dp\, (1+|p|)^{2\lambda}|\tilde{\chi}(p-k)\tilde{\chi}(p-k')| \equiv F_\delta(k,k') \tag{4.26}$$

Therefore, (4.25) is dominated by

$$\int_{-\pi/\delta}^{\pi/\delta} dk \int_{-\pi/\delta}^{\pi/\delta} dk' \{\mu(k)^{-1+\alpha/2}\mu(k')^{-1+\alpha/2} + \mu(k)^{-1}\mu(k')^{-1+\alpha}\}F_\delta(k, k')^2 \tag{4.27}$$

To estimate (4.27) we divide up the $k - k'$ integration region into four subregions:

Region 1: Both $k, k' \in T_{2\delta} = [-\pi/2\delta, \pi/2\delta]$.
Region 2: One of $k, k' \in T_{2\delta}$.
Region 3: Neither $k, k' \in T_{2\delta}$ with k, k' on opposite sides of $T_{2\delta}$.
Region 4: Neither $k, k' \in T_{2\delta}$ with k, k' on the same side of $T_{2\delta}$.

In *region 1* we have $|p - k|$, $|p - k'| \leqslant 3\pi/2\delta$ in (4.26), so that by Lemma 4.7

$$F_\delta(k, k') \leqslant c\int_{-\infty}^{\infty} dp\,(1 + |p|)^{2\lambda}(1 + |p - k|)^{-1}(1 + |p - k'|)^{-1}$$

By Lemma C3, for $\nu < 1 - 2\lambda$,

$$F_\delta(k, k') \leqslant c(1 + |k - k'|)^{-\nu}(1 + |k|^{2\lambda} + |k'|^{2\lambda}) \equiv F(k, k') \tag{4.28}$$

Plugging this estimate into (4.27), we see by Lemma C3 that the resulting integral is convergent, provided $2\nu > \alpha + 4\lambda$, i.e., if $4\lambda < 1 - \alpha/2$.

In *region 2* we may assume without loss of generality that $|k| \leqslant \pi/2\delta$ and $\pi/2\delta \leqslant k' \leqslant \pi/\delta$. We write

$$F_\delta(k, k') = c\left(\int_{-\pi/\delta}^{-3\pi/4\delta} dp + \int_{-3\pi/4\delta}^{\pi/\delta} dp\right)[(1 + |p|)^{2\lambda}|\tilde{\chi}(p - k)\tilde{\chi}(p - k')|] \tag{4.29}$$

In the second integral we still have $|p - k'| \leqslant 7\pi/4\delta$, so that by (a slight generalization of) Lemma 4.7 we carry out the estimates as in region 1. As for the first integral in (4.29), we bound it by Lemma 4.7:

$$\int_{-\pi/\delta}^{-3\pi/4\delta} dp\,(1 + |p|)^{2\lambda}(1 + |p - k|)^{-1}|\tilde{\chi}(p - k')| \leqslant c\delta^{-2\lambda}(1 + \pi/4\delta)^{-1}\int_{-\pi/\delta}^{\pi/\delta} dp\,|\tilde{\chi}(p - k')| \tag{4.30}$$

since $|p - k| \geqslant \pi/4\delta$. By the Schwarz inequality

$$\int_{-\pi/\delta}^{\pi/\delta} |\tilde{\chi}(p - k')|\,dp \leqslant c\delta^{-1/2}\left[\int |\tilde{\chi}(p - k')|^2\,dp\right]^{1/2} = O(\delta^{-1/2})$$

so that (4.30) is $O(\delta^{1-2\lambda-1/2})$. Plugging this bound into (4.27), we obtain a bound uniform in δ provided $1 - 2\lambda - 1/2 > \alpha/2$, i.e., if $4\lambda < 1 - \alpha$.

In *region 3* we may assume without loss of generality that $k \leqslant -\pi/2\delta$, $k' \geqslant \pi/2\delta$ and we may restrict ourselves to integration over $p \geqslant 0$ in (4.26). Since

$$0 \leqslant k - p + 2\pi/\delta \leqslant 3\pi/2\delta, \qquad -\pi/2\delta \leqslant k' - p \leqslant \pi/\delta$$

it follows as in the case of region 1 from the periodicity of $\tilde{\chi}$ that

$$\begin{aligned}\int_0^{\pi/\delta} & dp\,(1 + p)^{2\lambda}|\tilde{\chi}(p - k)\tilde{\chi}(p - k')| \\ &\leqslant c\int_{-\infty}^{\infty} dp\,(1 + |p|)^{2\lambda}(1 + |k' - p|)^{-1}(1 + |k + 2\pi/\delta - p|)^{-1} \\ &\leqslant cF(k + 2\pi/\delta, k')\end{aligned}$$

where F is defined in (4.28). Plugging this bound into (4.27) and using the fact that $|k + 2\pi/\delta| \leqslant c|k|$, we obtain a bound independent of δ by Lemma C3 provided $4\lambda < 1 - \alpha/2$.

In *region 4* we may assume that $\pi/2\delta \leqslant k, k' \leqslant \pi/\delta$. For $p \geqslant 0$, the estimate proceeds as in region 1. For $p \leqslant 0$ we have $p - k + 2\pi/\delta$ and $p - k' + 2\pi/\delta$ in $[0, 3\pi/2\delta]$, and we reason as in the case of region 3 to obtain a bound

$$cF(k + 2\pi/\delta, k' + 2\pi/\delta) \leqslant c'F(k, k')$$

This completes the argument that (4.27) and hence (4.25) are bounded uniformly in δ. ∎

Taking stock, we see that we have verified hypotheses (i) and (ii) of Theorem 2.1. As for hypothesis (iii):

Lemma 4.10. There is a constant c independent of δ and j such that

$$\int \mathrm{Tr}(K_{\delta,j}^2 + K_{\delta,j}^* K_{\delta,j})\, d\mu \leqslant cj \tag{4.31}$$

Proof. By (2.26) the left side of (4.31) equals

$$\int d\mu \iint_{\Lambda\times\Lambda} dx\, dy\, v(x - y)\phi_{\delta,j}(x)\phi_{\delta,j}(y) = \iint_{\Lambda\times\Lambda} dx\, dy\, v(x - y)C_{\delta,j}(x, y) \tag{4.32}$$

where

$$\begin{aligned}\hat{v}(p) &= c\int \left[\frac{m^2 - p\cdot(p + q)}{D(p)^2 D(p + q)^2} + \frac{1}{D(p)D(p + q)}\right] dq \\ &= O(1)\end{aligned} \tag{4.33}$$

by Appendix A of Ref. 16. Hence in momentum space

$$(4.32) = c\int dp \int_{T_\delta} dk\, \hat{v}(p)\mu_\delta(k)^{-2}e^{-\mu_\delta/\kappa}\psi(p)^2\tilde{\chi}(p-k)^2$$
$$\leqslant \int_{T_\delta} dk\, \mu_\delta^{-2}e^{-\mu_\delta/\kappa} \leqslant cj$$

by (4.33), (4.15), and (4.6). ■

The last two hypotheses of Theorem 2.1 concern the object

$$W_\delta = \tfrac{1}{2}{:}\mathrm{Tr}(K_\delta{}^\dagger K_\delta - K_\delta{}^* K_\delta){:}$$
$$= \iint_{\Lambda\times\Lambda} dx\, dy\, w(x-y){:}\phi_\delta(x)\phi_\delta(y){:}$$

where, as in the case of B_δ (Ref. 16, Appendix A),

$$\hat{w}(p) \leqslant c\log(2 + |p|) \tag{4.34}$$

Since the lattice cutoff does not affect the kernel w of W_δ, we have $\hat{w}(p) \geqslant 0$ (see the discussion before Theorem 2.2); i.e., hypothesis (v) of Theorem 2.1 holds. The remaining hypothesis (iv) follows by an identical argument to that of Lemma 4.9, since we have the same bound (4.34) for w as we had for b [see (4.24)]. Thus:

Lemma 4.11. $\|W_\delta\|_{L^2} < \infty$, uniformly in δ.

To summarize: we have verified the hypotheses of Theorem 2.1 for ρ_δ and have thus established Lemma 4.2. It remains to prove Lemmas 4.3 and 4.4.

Proof of Lemma 4.3. By complex interpolation [see Eq. (2.9)] and an application of the Schwarz inequality, it suffices to show for some $\epsilon > 0$ that

$$\|D^{-\epsilon}(K_\delta - K)\|_{2,2} \to 0 \tag{4.35}$$

$$\|D^{\epsilon}(K_\delta - K)\|_{4,4} \to 0 \tag{4.36}$$

as $\delta \to 0$. By a computation similar to the one in the proof of Lemma 4.6, we see that

$$\|D^{-\epsilon}(K_\delta - K)\|_{2,2}^2 = c\int dp \int dp'\, \mu(p)^{-1}\mu(p')^{-1-2\epsilon}\int_{T_\delta} dk \times \left|\frac{\hat{\chi}_\Lambda(p-p'+k)}{\mu(k)} - \frac{\psi(p-p')\tilde{\chi}(p-p'+k)}{\mu_\delta(k)}\right| \tag{4.37}$$

where we have applied both (4.4) and the fact that

$$\int \overline{\hat{\phi}(k)}\tilde{q}(k')\, d\mu = \delta(k-k')\mu_\delta(k)^{-1}\mu(k)^{-1}, \qquad k \in \mathbb{R}^2, \quad k' \in T_\delta$$

As in the case of $P(\phi)_2$ (Ref. 32, Lemma IX.4), the convergence to 0 of (4.37) is proven via the dominated convergence theorem. First, we make the change of variable $p \mapsto p + p'$ and appeal to Lemma C5 to integrate out p':

$$\|D^{-\epsilon}(K_\delta - K)\|_{2,2}^2 \leqslant c \iint dp\, dk\, \chi_{T_\delta}(k)\mu(p)^{-2\epsilon}\left|\frac{\hat{\chi}_\Lambda(p+k)}{\mu(k)} - \frac{\psi(p)\tilde{\chi}(p+k)}{\mu_\delta(k)}\right|^2 \tag{4.38}$$

Second, we note that as $\delta \to 0$, $\chi_{T_\delta}(k) \to 1$, $\mu_\delta(k) \to \mu(k)$,

$$\psi(p) \to 1 \tag{4.39a}$$

and

$$\tilde{\chi}(p + k) \to \hat{\chi}_\Lambda(p + k) \tag{4.39b}$$

for $p, k \in \mathbb{R}^2$, so the integrand of (4.38) converges to zero pointwise. [(4.39a) follows from (4.16c), and (4.39b) holds because (4.16a) is just the Riemann sum for the integral that gives $\hat{\chi}_\Lambda$.] Thus, we need only bound the integrand in (4.38) by a sum of two integrable functions, where one of them is independent of δ and the integral of the other one *goes to zero* as $\delta \to 0$. Now

$$|\hat{\chi}_\Lambda(p+k)| = \prod_{i=0}^{1} \frac{|\sin[(p_i + k_i)L/2]|}{|p_i + k_i|L/2} \leqslant \text{const} \prod_{i=0}^{1} (1 + |p_i + k_i|)^{-1}$$

so that it suffices to bound the function

$$\mu(p)^{-2\epsilon}\chi_{T_\delta}(k)\psi(p)^2\tilde{\chi}(p+k)^2\mu_\delta(k)^{-2}$$

As usual, it suffices to estimate the one-dimensional expression [we have also applied (4.8)]

$$\mu(p)^{-\epsilon}\chi_{T_\delta}(k)\psi(p)^2\tilde{\chi}(p+k)^2\mu(k)^{-1}$$

We break this up into two terms via $1 = \chi_{T_{2\delta}}(p) + [1 - \chi_{T_{2\delta}}(p)]$ and apply Lemma 4.7 to dominate the first term by

$$\text{const} \times \mu(p)^{-\epsilon}(1 + |p + k|)^{-2}\mu(k)^{-1}$$

which is both independent of δ and integrable. The integral of the second term is dominated by

$$\begin{aligned}
&\sum_{n\neq 0}\int_{-\pi/\delta}^{\pi/\delta} dk\, \mu(k)^{-1}\int_{-\pi/\delta}^{\pi/\delta} dp\, \psi(p - 2\pi n\delta^{-1})^2\tilde{\chi}(p+k)^2\mu(p)^{-\epsilon}\\
&\quad + \int_{-\pi/\delta}^{\pi/\delta} dk\, \mu(k)^{-1}\int_{1/2\pi\delta^{-1}\leq|p|\leq\pi\delta^{-1}} dp\, \tilde{\chi}(p+k)^2\psi(p)^2\mu(p)^{-\epsilon}\\
&\leqslant c\left(\sum_{n\neq 0}\frac{1}{n^2}\right)\int_{-\pi/\delta}^{\pi/\delta} \mu(k)^{-1-\epsilon/2}\, dk\, L\mu(\tfrac{1}{2}\pi\delta^{-1})^{-\epsilon/2}\\
&\quad + c\int_{-\pi/\delta}^{\pi/\delta} \mu(k)^{-1-\epsilon/2}\, dk\, L\mu(\tfrac{1}{2}\pi\delta^{-1})^{-\epsilon/2}\\
&\leqslant c\mu(\tfrac{1}{2}\pi\delta^{-1})^{-\epsilon/2}
\end{aligned}$$

where we have estimated as in the proof of Lemma 4.5. Obviously, $\mu(\frac{1}{2}\pi\delta^{-1})^{-\epsilon/2} \to 0$ as $\delta \to 0$.

We prove (4.36) by the estimate

$$\|D^{\epsilon}(K_{\delta} - K)\|_{4,4}^{4} \leqslant \text{const}\ \|D^{\epsilon}(K_{\delta} - K)D^{-\alpha}\|_{2,2}^{2}\|D^{\epsilon}(K_{\delta} - K)D^{\alpha}\|_{4,4}^{2}$$

[see (4.22)]. The first factor converges to zero for $\alpha > \epsilon$, by a trivial modification of the above argument. If $\alpha + \epsilon < \frac{1}{2}$, the second factor is bounded uniformly in δ by explicit computation in configuration space [see (4.21)]. ∎

Proof of Lemma 4.4. As in the computations of Lemma 4.9,

$$\begin{aligned}\int (B - B_{\delta})^{2}\, d\mu = \text{const} &\int_{T_{\delta}} dk \int_{T_{\delta}} dk' \int dp \int dp'\, \hat{b}(p)\hat{b}(p') \\ &\times \left[\frac{\hat{\chi}_{\Lambda}(k + p)\hat{\chi}_{\Lambda}(k' - p)}{\mu(k)\mu(k')} - \psi(p)^{2}\frac{\tilde{\chi}(k + p)\tilde{\chi}(k' - p)}{\mu_{\delta}(k')\mu_{\delta}(k)}\right] \\ &\times \left[\frac{\hat{\chi}_{\Lambda}(p' - k)\hat{\chi}_{\Lambda}(p' + k')}{\mu(k')\mu(k)} - \psi(p')^{2}\frac{\tilde{\chi}(p' - k)\tilde{\chi}(p' + k')}{\mu_{\delta}(k')\mu_{\delta}(k)}\right] \\ &+ \text{similar integral.}\end{aligned}$$

We concentrate on the first integral, which can be rewritten as

$$\int_{T_{\delta}^{2}} dk\, dk' \left| \int dp\, \hat{b}(p)[f(p, k, k') - f_{\delta}(p, k, k')] \right|^{2} \tag{4.40}$$

where

$$f(p, k, k') = \frac{\hat{\chi}_{\Lambda}(p + k)\hat{\chi}_{\Lambda}(p - k')}{\mu(k)\mu(k')}$$

$$f_{\delta}(p, k, k') = \psi(p)^{2}\frac{\tilde{\chi}(p + k)\tilde{\chi}(p - k')}{\mu_{\delta}(k)\mu_{\delta}(k')}$$

because $\hat{b}$, $\hat{\chi}_{\Lambda}$, $\tilde{\chi}$, and ψ are even functions. Clearly, as $\delta \to 0$,

$$f_{\delta}(p, k, k') \to f(p, k, k')$$

so to show that $\int d^{2}p\, \hat{b}(p)[f(p, k, k') - f_{\delta}(p, k, k')] \to 0$ it suffices to show that

$$\psi(p)^{2}\tilde{\chi}(p + k)\tilde{\chi}(p - k')\hat{b}(p)$$

is dominated by a sum of p-integrable functions that are either independent of δ or have p-integrals converging to zero as $\delta \to 0$. We see that this can be done using the estimations in the proof of Lemma 4.9. In those cases where one does not obtain an integrable bound independent of δ, one can always pick up a factor of $\mu(\delta^{-1})^{-\epsilon}$, which goes to zero as $\delta \to 0$. We omit the details.

To complete the proof that (4.40) $\to 0$ it then suffices to bound

$$\chi_{T_\delta}(k)\chi_{T_\delta}(k')\left| \int dp\, \hat{b}(p) f_\delta(p, k, k') \right|^2$$

by a sum of two integrable functions of k, k' where one of them is independent of δ and the integral of the other goes to zero. Such a bound can be extracted from the proof of Lemma 4.9. ∎

This completes our proof of Theorem 4.1.

5. SPACE CUTOFF OF THE FERMI MASS

Throughout this section the lattice cutoff on the boson field is in effect. Thus, in terms of the Fermi two-point function S of (1.12), we set

$$K = S\phi_\delta \chi_\Lambda, \qquad S' = (1 - K)^{-1} S \tag{5.1}$$

regarded as operators on $\mathscr{H} = \mathscr{H}_m$. The corresponding objects with a spatially cutoff Fermi mass in χ_{D_R} are defined as the operators on $\mathscr{H}_0$:

$$S_R = (1 + S_0 m \chi_{D_R})^{-1} S_0 \tag{5.2a}$$

$$K_R = S_R \phi_\delta \chi_\Lambda \tag{5.2b}$$

$$S_R' = (1 - K_R)^{-1} S_R = (1 - S_0 \phi_\delta \chi_\Lambda + S_0 m \chi_{D_R})^{-1} S_0 = (1 - S_0 h)^{-1} S_0 \tag{5.2c}$$

where $h = \phi_\delta \chi_\Lambda - m\chi_{D_R}$. Referring to (3.1), we see that by Theorem 3.3 we have an inequality of the desired form (1.19), namely,

$$\operatorname{tr} S_R'(x, y) S_R'(y, x) < 0, \qquad x \neq y \tag{5.3}$$

However, it remains to remove the spatial cutoff, i.e., to take the limit $R \to \infty$.

It is tempting to try to take $R \to \infty$ directly in (5.3). Indeed, one can obtain an interesting result along these lines:

Theorem 5.1. If $(1 - K)^{-1}$ is a bounded operator on $\mathscr{H}_m$, then for almost all x, y there is a sequence $\{R_n\}$ such that $R_n \to \infty$ and $\lim_{n\to\infty} S_{R_n}'(x, y) = S'(x, y)$ and hence

$$\operatorname{tr} S'(x, y) S'(y, x) \leqslant 0 \tag{5.4}$$

for almost all x, y.

We give a proof of this theorem in Appendix B. The trouble with this approach, however, is that we do not know whether $(1 - K)^{-1} = (1 - S\phi_\delta\chi_\Lambda)^{-1}$ is bounded for all possible values of the lattice field ϕ_δ. Equivalently, we do not know whether $\det_3(1 - S\phi_\delta\chi_\Lambda) > 0$ for all ϕ_δ (see Ref. 16 for some results in this direction). In the application of Theorem 1.1 to the proof of the FKG inequality it seems crucial that the interaction density involved does not vanish *anywhere* (see Appendix A). Therefore we are obliged to consider the convergence of the *entire* theory as $R \to \infty$ and not just that of the two-point function (see Theorem 5.2 below).

We first face a slight problem in that the operators in (5.1) and (5.2) are defined on different Hilbert spaces, $\mathscr{H}$ and $\mathscr{H}_0$, respectively. By Theorem 3.1, K_R is a $\mathscr{C}_4$ operator on $\mathscr{H}_0$ and $(1 - K_R)^{-1}$ is bounded on $\mathscr{H}_0$. But K_R is not an operator on $\mathscr{H}$, because its image elements decay too slowly at infinity in x space. To remedy this discrepancy we find it convenient to introduce an additional spatial cutoff: let $\zeta \in C_0^\infty$ with supp $\zeta \subset D_1$ and $\zeta = 1$ on $D_{1/2}$, and let $\zeta_R(x) = \zeta(x/R)$, so that supp $\zeta_R \subset D_R$ and $\zeta_R = 1$ on $D_{R/2}$. Since we are interested in the limiting situation as $R \to \infty$, we may assume that $\Lambda \subset D_{R/4}$; from the definition (5.2b) we have

$$K_R\zeta_R = K_R \tag{5.5}$$

Now by Lemma C7, $\zeta_R K_R$ is a bounded operator on $\mathscr{H}$. In fact, by the remark following Lemma 5.5 below, $\zeta_R K_R$ is $\mathscr{C}_3$ on $\mathscr{H}$. Hence it makes sense to consider $\det_3(1 - \zeta_R K_R)$ on $\mathscr{H}$. We therefore take as our cutoff density [see (1.9), (1.16)]

$$\rho_R = \det_3(1 - \zeta_R K_R) \exp\{-\tfrac{1}{2}{:}\mathrm{Tr}[(\zeta_R K_R)^2 + K^\dagger K]{:}\} \tag{5.6}$$

Note that (formally) the ζ_R can actually be dropped from (5.6) because of (5.5) and the relation $\det(1 - AB) = \det(1 - BA)$. The noncutoff density is

$$\rho = \det_3(1 - K) \exp[-\tfrac{1}{2}{:}\mathrm{Tr}(K^2 + K^\dagger K){:}]$$

Our goal in this section is to prove:

Theorem 5.2. For any $p < \infty$, $\rho_{R_n} \to \rho$ in $L^p(d\mu)$ for some sequence $R_n \to \infty$.

This convergence result is more involved than that of Theorem 5.1, but it avoids the problem of the vanishing determinant, because the correlation inequality for the case of cutoff mass will now merely carry over to the limiting case of constant mass. The following lemma establishes positivity of the determinant in the cutoff mass case.

Lemma 5.3. (a) $(1 - \zeta_R K_R)^{-1}$ is a bounded operator on $\mathscr{H}$ and

$$(1 - \zeta_R K_R)^{-1} = 1 - \zeta_R + \zeta_R(1 - K_R)^{-1} \tag{5.7}$$

(b) For all ϕ_δ, $\det_3(1 - \zeta_R K_R) > 0$.

Proof. (a) Since $(1 - K_R)^{-1}$ is a bounded operator on $\mathscr{H}_0$, it follows from Lemma C7 that $\zeta_R(1 - K_R)^{-1}$ is a bounded operator on $\mathscr{H}$. By (5.5)

$$\begin{aligned}(1 - \zeta_R K_R)&[1 - \zeta_R + \zeta_R(1 - K_R)^{-1}]\\ &= 1 - \zeta_R + \zeta_R(1 - K_R)^{-1} - \zeta_R K_R(1 - K_R)^{-1}\\ &= 1\end{aligned}$$

The computation of the product in the reverse order is similar. This establishes (5.7) and part (a), since multiplication by ζ_R is a bounded operator on $\mathscr{H}$.

(b) Since $\zeta_R K_R$ has a real kernel, $\det_3(1 - \zeta_R K_R)$ is real. By part (a) we know that $\det_3(1 - \lambda\zeta_R K_R) \neq 0$ for any $\lambda \in \mathbb{R}$ and that it is a continuous function of λ. At $\lambda = 0$ it has the value 1 and hence must be positive everywhere. ■

Our proof of Theorem 5.2 follows the strategy of Lemma 2.4 and so we begin by showing:

Lemma 5.4. For any $p < \infty$, $\rho_R \in L^p(d\mu)$ uniformly in R.

Proof. We write

$$\zeta_R K_R = K + (\zeta_R K_R - K) \equiv K + V_R \tag{5.8}$$

and apply Theorem 2.2 to $K' = \zeta_R K_R$. Actually in the present situation with a fixed lattice cutoff $\delta > 0$, we do not need the full force of Theorem 2.2 and we can dispense with the expansion of ρ_R in terms of a sequence of ultraviolet cutoffs. We need only verify that hypotheses (b) and (d) of Theorem 2.2 hold with estimates uniform in R. But hypothesis (b) on K has already been checked in the paragraph preceding Theorem 2.2, and the bounds are certainly uniform in R since K is independent of R. It thus remains to verify (d): for some $\epsilon > 0$,

$$\|D^\epsilon V_R\|_{2,2} + \|V_R D^\epsilon\|_{2,2} < \infty \tag{5.9}$$

uniformly in R. It is obvious that (5.9) is implied by the next lemma. ■

Lemma 5.5 For $\epsilon < 1/2$

$$\|D^\epsilon V_R\|_{2,2} + \|V_R D^\epsilon\|_{2,2} \to 0 \tag{5.10}$$

as $R \to \infty$.

Remark. The proof of the lemma shows that $D^\epsilon V_R \in \mathscr{C}_2 \subset \mathscr{C}_3$ for all ϕ_δ. Since $K \in \mathscr{C}_3$, we see that $\zeta_R K_R \in \mathscr{C}_3$.

The proof of Lemma 5.5 is based on an explicit calculation of S_R that provides a bound uniform in R. Such a calculation is possible because the spherical symmetry of the cutoff mass allows a series expansion for S_R involving Bessel functions. We shall use complex notation as in Section 3: let $z = x_0 + ix_1$, let $t = y_0 + iy_1 \in \Lambda$ be fixed for the moment and set

$$u(z) = -S_R^{01}(x, y) - iS_R^{11}(x, y), \qquad \tilde{u}(z) = S_R^{00}(x, y) + iS_R^{10}(x, y)$$

bearing in mind that u and $\tilde{u}$ will depend on both R and t. Then we have the integral equations

$$u(z) - \frac{im}{\pi}\int_{|\zeta| \le R} d^2\zeta\, \frac{\overline{u(\zeta)}}{z - \zeta} = \frac{1}{\pi(z - t)} \tag{5.11a}$$

$$\tilde{u}(z) - \frac{im}{\pi}\int_{|\zeta| \le R} d^2\zeta\, \frac{\overline{\tilde{u}(\zeta)}}{z - \zeta} = \frac{-i}{\pi(z - t)} \tag{5.11b}$$

We will concentrate on (5.11a), since the solution of (5.11b) will be similar. The differential equation corresponding to (5.11a) is

$$\partial_{\bar{z}} u(z) - im\chi_{D_R}(z)\overline{u(z)} = \delta(z - t)$$

Let w be the standard solution of the free equation

$$\partial_{\bar{z}} w(z) - im\overline{w(z)} = \delta(z - t)$$

and set $v = u - w$. Thus we have the equations

$$\begin{aligned} \partial_{\bar{z}} v(z) - im\overline{v(z)} &= 0, \qquad |z| < R \\ \partial_{\bar{z}} u(z) &= 0, \qquad |z| > R \end{aligned}$$

and $u(z)$ must agree with $v(z) + w(z)$ at $|z| = R$. Since $u(z)$ is analytic for $|z| > R$ and must decay like z^{-1}, we have the expansion

$$u(z) = \sum_{m=1}^{\infty} c_{-m} z^{-m}, \qquad |z| > R$$

Transforming to polar coordinates, we get

$$\begin{aligned} e^{i\theta}\frac{\partial v}{\partial r} + \frac{ie^{i\theta}}{r}\frac{\partial v}{\partial \theta} - im\bar{v} &= 0, \qquad r < R \\ u = \sum_{m=1}^{\infty} c_{-m} r^{-m} e^{-im\theta}&, \qquad r > R \end{aligned} \tag{5.12}$$

Let $v_n(r)$ and $w_n(r)$ be the Fourier coefficients with respect to θ of v and w, respectively. Then (5.12) becomes

$$v_n'(r) - \frac{n}{r} v_n(r) - im\overline{v_{-n-1}(r)} = 0, \qquad r < R \tag{5.13}$$

Combining this equation with the one obtained by the index change $n \to -n - 1$, we get the second-order equation

$$v_n''(r) + \frac{1}{r} v_n'(r) - \left(m^2 + \frac{n^2}{r^2}\right) v_n(r) = 0$$

This is the equation satisfied by the nth-order Bessel functions with imaginary argument. Thus (see Ref. 34) for the definitions and properties of the Bessel functions we use)

$$v_n(r) = a_n I_n(mr), \qquad r < R$$

and

$$u(z) = w(z) + \sum a_n I_n(mr) e^{in\theta}, \qquad r = |z| < R \tag{5.14}$$

[$K_n(mr)$ does not appear because v_n is regular on the disk.] Applying the recursion relations

$$I_{n-1}(mr) - I_{n+1}(mr) = (2n/mr)I_n(mr) \qquad (5.15)$$
$$I_{n-1}(mr) + I_{n+1}(mr) = 2I_n{}'(mr)$$

and the symmetry $I_{-n} = I_n$, we find that Eq. (5.13) reduces to

$$a_{-n-1} = i\bar{a}_n$$

By the compatibility condition at $|z| = R$

$$a_n I_n(mR) + w_n(R) = 0, \qquad n \geqslant 0$$

so

$$a_n = -w_n(R)/I_n(mR), \qquad n \geqslant 0$$
$$a_n = -i\overline{w_{-n-1}(R)}/I_{n+1}(mR), \qquad n \leqslant -1$$

Now, $I_n(mr)$ is increasing in r, so that for $r < R$ and $n \geqslant 0$

$$|a_n| I_n(mr) \leqslant |w_n(R)| \qquad (5.16a)$$

For $n \leqslant -1$, it follows from (5.15) that

$$I_{n+1}(mR) \geqslant (2|n|/mR)I_n(mR)$$

so that

$$|a_n| I_n(mr) \leqslant (mR/2|n|)|w_{-n-1}(R)| \qquad (5.16b)$$

But w is just the second column of the two-point function (1.12), which is C^∞ for $z \neq t$ and together with its derivatives has exponential decay in $|z - t|$. Hence by integration by parts for $n > 0$

$$|w_n(R)| \leqslant \frac{1}{n^2}\int_0^{2\pi} \left|\frac{\partial^2 w(Re^{i\theta})}{\partial\theta^2}\right| d\theta \leqslant \frac{c}{n^2}\, e^{-a(R-|t|)}$$

where $0 < a < m$; the case $n < 0$ is similar. In this way we obtain from (5.16), for $r < R$,

$$|a_n| I_n(mr) \leqslant [c/(n^2 + 1)]e^{-a(R-|t|)}$$

Substitution of this bound into (5.14) yields for $R/2 \leqslant |z| \leqslant R$

$$|u(z)| \leqslant |w(z)| + ce^{-a(R-|t|)} \leqslant \text{const}\, e^{-a|z-t|} \qquad (5.17)$$

As the calculation for $\tilde{u}(z)$ is practically identical, we may conclude that (5.17) holds for the components of S_R:

Lemma 5.6. There are constants $a > 0$ and $c > 0$ such that for $R/2 \leqslant |x| \leqslant R$ and $|y| \leqslant R/4$,

$$|S_R^{\alpha\beta}(x, y)| \leqslant ce^{-a|x-y|} \qquad (5.18)$$

Proof of Lemma 5.5. Since S_R satisfies the equation [see (1.13)]

$$[-\beta\cdot\partial_x + m\chi_{D_R}(x)]S_R(x, y) = \delta(x - y)$$

we have

$$\begin{aligned}(-\beta\cdot\partial_x + m)&[\zeta_R(x)S_R(x, y) - S(x, y)]\\ &= (-\beta\cdot\partial\zeta_R)S_R + \zeta_R m(1 - \chi_{DR})S_R + (\zeta_R - 1)\delta = (-\beta\cdot\partial\zeta_R)S_R\end{aligned}$$

if $y \in \Lambda$, since $\zeta_R = 0$ outside D_R and $\zeta_R = 1$ on Λ. Let $h_R = -\beta\partial\zeta_R$. Thus, from the definition (5.8) of V_R, $(-\beta\partial + m)V_R = h_R K_R$. We then compute that

$$\begin{aligned}\|D^\epsilon V_R\|_2{}^2 &= \mathrm{Tr}\ D^{-1}V_R{}^\dagger D^{1+2\epsilon}V_R\\ &= \mathrm{Tr}\ D^{-1}[(-\beta\partial + m)V_R]^\dagger D^{-1+2\epsilon}(-\beta\partial + m)V_R\\ &= \mathrm{Tr}\ D^{-1}K_R{}^\dagger h_R D^{-1+2\epsilon}h_R K_R\\ &= \iint_{\Lambda\times\Lambda} dx\, dy\, D^{-1}(x - y)\phi_\delta(x)\phi_\delta(y)\\ &\quad\times \left[\iint dz\, dz'\ \mathrm{tr}\ S_R(y, z)h_R(z)D^{-1+2\epsilon}(z - z')h_R(z')S_R(z', x)\right]\end{aligned} \tag{5.19}$$

Since $h_R(z)$ is supported in the annulus $D_R\backslash D_{R/2}$, we can apply the bound (5.18) to conclude that the quantity in square brackets in (5.19) is $O(e^{-aR/2})$. Note that $h_R(z) = O(R^{-1})$ and that $D^{-1+2\epsilon}$ has an integrable singularity for $\epsilon < 1/2$. Hence $\|D^\epsilon V_R\|_2{}^2 = O(e^{-aR/2})$. The bound on $V_R D^\epsilon$ is similar and we obtain (5.10) upon integrating with respect to $d\mu$. ■

Proof of Theorem 5.2. By Lemmas 2.4 and 5.4, the proof of the theorem will be completed by the facts

$$\|\zeta_R K_R - K\|_{3,3} \to 0 \tag{5.20}$$

$$\|:\mathrm{Tr}[(\zeta_R K_R)^2 - K^2]:\|_{L^2} \to 0 \tag{5.21}$$

as $R \to \infty$. But by hypercontractivity,[22]

$$\|\zeta_R K_R - K\|_{3,3} = \|V_R\|_{3,3} \leqslant \text{const} \times \|V_R\|_{2,2}$$

so that (5.20) is a consequence of (5.10). As for (5.21), note that

$$\begin{aligned}|\mathrm{Tr}[(\zeta_R K_R)^2 - K^2]| &= |\mathrm{Tr}\ V_R{}^2 + 2\,\mathrm{Tr}\ V_R K|\\ &\leqslant \|V_R\|_2{}^2 + 2\|V_R D^{2\epsilon}\|_2\|D^{-2\epsilon}K\|_2\end{aligned}$$

Hence (5.21) also follows from (5.10). ■

6. FKG INEQUALITY FOR SCALAR Y_2

We are now ready to prove our main result, namely, that (1.28) holds. As we have already shown in the introduction, this condition yields the FKG inequality for the lattice spin system that we have when both the lattice cutoff and the space cutoff of Fermi mass have been imposed on the scalar Y_2 model. We have also seen how Theorem 1.2 follows from this result together with the convergence theorems of Sections 4 and 5. As our formal calculation with variational derivatives in the introduction has indicated, the main input in the proof of (1.28) will be the theorems of Section 3.

In the cutoff model the (unnormalized) measure is

$$\rho(\tilde{K}, K)\, d\mu = \det_3(1 - \tilde{K}) \exp[-\tfrac{1}{2}{:}\mathrm{Tr}(\tilde{K}^2 + K^\dagger K){:}]\, d\mu \tag{6.1}$$

where $d\mu$ now denotes the free boson measure restricted to the lattice variables $q_1,..., q_N$ associated to the lattice squares $\Delta_1,..., \Delta_N$, respectively, where, in this section, $\Delta_1,..., \Delta_N$ will denote those lattice squares that lie in the box Λ. Thus

$$\chi_\Lambda \phi_\delta = \sum_{j=1}^{N} \chi_j q_j$$

where χ_j denotes the characteristic function of Δ_j. K is defined by (5.1) and

$$\tilde{K} \equiv \zeta_R K_R = \zeta_R S_R \phi_\delta \chi_\Lambda \tag{6.2}$$

where S_R is defined by (5.2a) and $\zeta_R \in C_0^\infty(\mathbb{R}^2)$ is specified in Section 5. We also set $\tilde{S} = \zeta_R S_R$, so that $\tilde{K} = \tilde{S}\phi_\delta\chi_\Lambda$.

We wish to show that the measure (6.1) satisfies the hypotheses of Theorem 1.1. There is nothing to prove about $d\mu$ since by the very definition of the lattice approximation[5] $d\mu = \text{const} \times \exp(-\frac{1}{2}q\cdot Aq)\, d^N q$ where A is an $N \times N$, positive-definite matrix such that, for $i \neq j$, $(\partial^2/\partial q_i\, \partial q_j)(-\frac{1}{2}q\cdot Aq) = -A_{ij} \geqslant 0$. It thus remains to show that

$$W \equiv \log \rho(\tilde{K}, K) = \log \det_3(1 - \tilde{K}) - \tfrac{1}{2}{:}\mathrm{Tr}(\tilde{K}^2 + K^\dagger K){:} \tag{6.3}$$

is in $C^2(\mathbb{R}^N)$ and that

$$\partial^2 W/\partial q_j\, \partial q_k \geqslant 0, \qquad j \neq k \tag{6.4}$$

Notice that W makes sense *for all* q because $\det_3(1 - \tilde{K})$ is strictly positive by Lemma 5.3. Moreover, $U \equiv (1 - \tilde{K})^{-1}$ is a bounded operator on $\mathscr{H}_m$. We analyze the two terms in (6.3) separately in the next two lemmas.

Lemma 6.1. $\log \det_3(1 - \tilde{K}) \in C^2(\mathbb{R}^N)$ and

$$\frac{\partial^2}{\partial q_j\, \partial q_k} \log \det_3(1 - \tilde{K}) = -\mathrm{Tr}(U\tilde{S}\chi_j U\tilde{K}^2\tilde{S}\chi_k) - \mathrm{Tr}(U\tilde{K}\tilde{S}\chi_j\tilde{S}\chi_k) - \mathrm{Tr}(U\tilde{S}\chi_j\tilde{K}\tilde{S}\chi_k) \tag{6.5}$$

Proof. From the definition (6.2), $\partial\tilde{K}/\partial q_k = \tilde{S}\chi_k$. Hence by Lemma 23 on p. 1110 of Ref. 24

$$\frac{\partial}{\partial q_k}\log \det_3(1 - \tilde{K}) = -\mathrm{Tr}(U\tilde{K}^2\tilde{S}\chi_k) \tag{6.6}$$

Note that the right side of (6.6) is well-defined since $\tilde{K}$ and $\tilde{S}\chi_k$ are in $\mathscr{C}_3$. We differentiate (6.6) again with respect to q_j: the resolvent identity

$$h^{-1}(U(q + hq_j) - U(q)) = -U(q + hq_j)\tilde{S}\chi_j U(q)$$

and the norm continuity of $U(q + hq_j)$ for $|h| < \|U(q)\tilde{S}\chi_j\|^{-1}$ show that

$$\partial U/\partial q_j = -U\tilde{S}\chi_j U$$

with convergence of the difference quotient in $\mathscr{C}_3$. Therefore by Hölder's inequality, (6.6) is differentiable with derivative given by (6.5). ■

Lemma 6.2. $:\mathrm{Tr}(\tilde{K}^2 + K^\dagger K): \in C^2(\mathbb{R}^N)$ and

$$\frac{\partial^2}{\partial q_j\,\partial q_k}\tfrac{1}{2}:\mathrm{Tr}(\tilde{K}^2 + K^\dagger K):$$
$$= \mathrm{Tr}(\tilde{S} - S)\chi_j\tilde{S}\chi_k + \mathrm{Tr}\, S\chi_j(\tilde{S} - S)\chi_k + \int_{\Delta_j}\int_{\Delta_k} dx\, dy\, b(x - y) \tag{6.7}$$

where $b(x - y)$ is the kernel of $:\mathrm{Tr}(K^2 + K^\dagger K):$.

Remark. For $x \neq y$ the distribution

$$b(x - y) = \mathrm{tr}\, S(x, y)S(y, x) \tag{6.8}$$

and in momentum space [see (2.26)]

$$\hat{b}(k) = \frac{2}{(2\pi)^3}\int\left[\frac{1}{D(p)^2} + \frac{m_f{}^2 - p\cdot(p + k)}{D(p)^2D(p + k)^2}\right]dp = O[\ln(2 + |k|)]$$

Hence the second term in (6.7) is well-defined by its momentum integral

$$2\pi\int dk\,\overline{\hat{\chi}_j(k)}\hat{b}(k)\hat{\chi}_k(k)$$

Proof. In the L^p proof for $\rho(\tilde{K}, K)$ in Section 5 we interpreted the expression $:\mathrm{Tr}(\tilde{K}^2 + K^\dagger K):$ as the sum of two well-defined terms

$$:\mathrm{Tr}(\tilde{K}^2 - K^2): + :\mathrm{Tr}(K^2 + K^\dagger K): \tag{6.9}$$

Explicitly, the first term equals

$$\sum_{j,k}[\mathrm{Tr}(\tilde{S} - S)\chi_j(\tilde{S} + S)\chi_k]:q_jq_k: \tag{6.10}$$

where the operator in question is trace-class by virtue of the estimate for $0 < \epsilon < \frac{1}{2}$,

$$\|(\tilde{S} - S)\chi_j D^\epsilon\|_2 + \|D^{-\epsilon}\tilde{S}\chi_j\|_2 + \|D^{-\epsilon}S\chi_j\|_2 < \infty \tag{6.11}$$

(this was essentially established in Lemma 5.5). The second term in (6.9) equals

$$\sum_{j,k}\iint dx\, dy\, \chi_j(x)b(x-y)\chi_k(y){:}q_jq_k{:} \tag{6.12}$$

Note that the Wick constants in (6.10) and (6.12) are finite because of the lattice cutoff. Differentiation of (6.9) yields the lemma. ■

To reproduce the (formal) calculations of Section 1 we need to manipulate the traces in (6.5) and (6.7) in a way that involves taking the trace of non-trace-class operators. Accordingly, we first regularize by replacing each χ_j (and χ_k) in (6.5) and (6.7) by the Hilbert–Schmidt operator $\chi_{j,n} \equiv \chi_j h_n{*}$, where $h_n{*}$ denotes convolution with $h_n(x) \equiv n^2 h(nx)$, where $h \in C_0^\infty(\mathbb{R}^2)$, $h \geqslant 0$, $\int h\, dx = 1$. The justification for this regularization is provided by the following lemma:

Lemma 6.3. (a) Suppose $f(x, y) \in L^p(\mathbb{R}^4)$, $1 \leqslant p \leqslant \infty$. Let

$$f * h_n(x, y) = \int f(x, x')h_n(x' - y)\, dx'$$

Then $f * h_n \to f$ in L^p as $n \to \infty$.

(b) Suppose $A \in \mathscr{C}_p(\mathscr{H})$, $1 \leqslant p \leqslant \infty$. Then $Ah_n{*} \to A$ in $\mathscr{C}_p$ as $n \to \infty$.

Proof. (a) This is Lemma VII.9(a) of Ref. 10; the proof uses the fact that, as an operator from L^p to L^p, $h_n{*}$ is bounded by $\|h_n\|_{L^1} = 1$.

(b) Note that $\hat{h}_n(p) = \hat{h}(p/n) \to 1/(2\pi)$ as $n \to \infty$.

For $p = 2$ we have explicitly

$$\|Ah_n{*} - A\|_2^2 = \text{const} \times \int D(p)D(q)^{-1}\left|\hat{A}(p,q)\left(\hat{h}_n(q) - \frac{1}{2\pi}\right)\right|^2 dp\, dq$$

which goes to 0 by the Lebesgue dominated convergence theorem. For $p < 2$ the result follows by noting that $h_n{*}$ is uniformly bounded in operator norm on $\mathscr{H}$ and approximating A in $\mathscr{C}_p$ norm by a finite rank operator A_0 and using the fact that by Hölder's inequality

$$\|A_0h_n{*} - A_0\|_p \leqslant N^{1/p - 1/2}\|A_0h_n{*} - A_0\|_2$$

where N is the rank of $A_0h_n - A_0$. For $p > 2$ we merely approximate by a $\mathscr{C}_2$ operator. ■

By Lemma 6.3b, $\tilde{S}\chi_{j,n} \to \tilde{S}\chi_j$ in $\mathscr{C}_3$, so that the traces in (6.5) converge as the regularization is removed. The same is true in (6.7) since, e.g.,

$$\mathrm{Tr}(\tilde{S} - S)\chi_{j,n}\tilde{S}\chi_{k,n} = \mathrm{Tr}[(\tilde{S} - S)\chi_j D^{\epsilon}]h_n^*(D^{-\epsilon}\tilde{S}\chi_k)h_n^*$$

which converges appropriately by (6.11) and the lemma.

By use of the identities

$$U\tilde{K} = U - 1, \qquad U\tilde{K}^2 = U - 1 - \tilde{K}$$

we then rewrite the (regularized) expression (6.5) as

$$\begin{aligned} &-[\mathrm{Tr}\ U\tilde{S}\chi_{j,n}U\tilde{S}\chi_{k,n} - \mathrm{Tr}\ U\tilde{S}\chi_{j,n}\tilde{S}\chi_{k,n} - \mathrm{Tr}\ U\tilde{S}\chi_{j,n}\tilde{K}\tilde{S}\chi_{k,n}] \\ &\qquad - [\mathrm{Tr}\ U\tilde{S}\chi_{j,n}\tilde{S}\chi_{k,n}) - \mathrm{Tr}(\tilde{S}\chi_{j,n}\tilde{S}\chi_{k,n})] - \mathrm{Tr}(U\tilde{S}\chi_{j,n}\tilde{K}\tilde{S}\chi_{k,n}) \\ &\qquad = -\mathrm{Tr}\ U\tilde{S}\chi_{j,n}U\tilde{S}\chi_{k,n} + \mathrm{Tr}\ \tilde{S}\chi_{j,n}\tilde{S}\chi_{k,n} \end{aligned} \tag{6.13}$$

Note that these calculations make sense since $\chi_{j,n} \in \mathscr{C}_2$. Similarly the first two terms in (6.7) may be written

$$\mathrm{Tr}\ \tilde{S}\chi_{j,n}\tilde{S}\chi_{k,n} - \mathrm{Tr}\ S\chi_{j,n}S\chi_{k,n} \tag{6.14}$$

From (6.3), (6.5), (6.7), (6.13), and (6.14) we obtain for the regularized second partial of W

$$\begin{aligned} \left(\frac{\partial^2 W}{\partial q_j\, \partial q_k}\right)_n &= -\mathrm{Tr}\ U\tilde{S}\chi_{j,n}U\tilde{S}\chi_{k,n} + \mathrm{Tr}\ S\chi_{j,n}S\chi_{k,n} - \int_{\Delta_j}\int_{\Delta_k} dx\, dy\, b(x-y) \\ &= \iint dx\, dy\, \mathrm{tr}[-(U\tilde{S}\chi_{j,n})(y,x)U\tilde{S}\chi_{k,n}(x,y) \\ &\qquad + (S\chi_{j,n})(y,x)(S\chi_{k,n})(x,y)] - \int_{\Delta_j}\int_{\Delta_k} dx\, dy\, b(x-y) \end{aligned} \tag{6.15}$$

We are justified in writing the trace as an integral over the diagonal of the kernel, since a product of two HS operators is involved (see Ref. 35, p. 522).

Now, since $K_R\zeta_R = K_R$, $(1 - \zeta_R K_R)^{-1}\zeta_R = \zeta_R(1 - K_R)^{-1}$; hence

$$\begin{aligned} \hat{S} \equiv U\tilde{S} &= (1 - \zeta_R K_R)^{-1}\zeta_R S_R \\ &= \zeta_R(1 - K_R)^{-1}S_R \\ &= \zeta_R[1 - (1 - S_0 m\chi_{D_R})^{-1}S_0\phi_\delta\chi_\Lambda]^{-1}(1 - S_0 m\chi_{D_R})^{-1}S_0 \\ &= \zeta_R(1 - S_0 h)^{-1}S_0 \end{aligned} \tag{6.16}$$

where $h = m\chi_{D_R} + \phi_\delta\chi_\Lambda$.

Lemma 6.4. $\hat{S}(x, y) - S(x, y) = O(\ln|x - y|^{-1})$ as $x \to y$.

Remark. We mean that each matrix element of $\hat{S} - S$ is $O(\ln|x - y|^{-1})$.

Proof. As in the proof of Lemma 5.5,

$$(-\not\nabla + m)(\hat{S} - S) = (-\not\nabla\zeta_R)\hat{S} + \zeta_R(m - h)\hat{S}$$

so that

$$\hat{S} - S = S(-\not\nabla\zeta_R)\hat{S} + S\zeta_R(m - h)\hat{S}$$

The lemma then follows from the known singular behavior

$$S(x, y) = O(|x - y|^{-1}), \qquad \hat{S}(x, y) = O(|x - y|^{-1}) \tag{6.17}$$

see Lemma 3.4) and Lemma C6. ■

We write the first term in (6.15) as

$$\iint_{\mathcal{N}} dx\, dy\, \mathrm{tr}[(S - \hat{S})\chi_j{*}h_n(y, x)S\chi_k{*}h_n(x, y) + \hat{S}\chi_j{*}h_n(y, x)(S - \hat{S})\chi_k{*}h_n(x, y)] \tag{6.18}$$

where the integration in (6.18) extends over a bounded neighborhood $\mathcal{N}$ of $\Delta_j \times \Delta_k$ (due to the smearing by h_n). By (6.17), $S\chi_k$ and $\hat{S}\chi_j \in L^p(\mathcal{N})$ for $p < 2$ and $(S - \hat{S})\chi_j$ and $(S - \hat{S})\chi_k \in L^q(\mathcal{N})$ for $q < \infty$. By Lemma 6.3(a) the convolutions with respect to h_n converge appropriately in L^p and L^q as $n \to \infty$, and so by Hölder's inequality, (6.18) converges to

$$\iint dx\, dy\, \mathrm{tr}[(S - \hat{S})\chi_j(y, x)S\chi_k(x, y) + \hat{S}\chi_j(y, x)(S - \hat{S})\chi_k(x, y)] \tag{6.19}$$

Now assume that $j \neq k$. Then it is not hard to show that

$$\int_{\Delta_j}\int_{\Delta_k} dx\, dy \frac{1}{|x - y|^2} < \infty$$

Therefore it makes sense to separate S and $\hat{S}$ in (6.19) to obtain

$$-\int_{\Delta_j}\int_{\Delta_k} dx\, dy\, \mathrm{tr}\, \hat{S}(y, x)\hat{S}(x, y) + \int_{\Delta_j}\int_{\Delta_k} dx\, dy\, \mathrm{tr}\, S(y, x)S(x, y) \tag{6.20}$$

Actually, since $\zeta_R\chi_j = \chi_j$, we can drop the ζ_R in (6.16) and replace $\hat{S}$ by $S' \equiv (1 - S_0 h)^{-1}S_0$. Moreover, from (6.8) when $j \neq k$,

$$\int_{\Delta_j}\int_{\Delta_k} b(x - y) = \int_{\Delta_j}\int_{\Delta_k} \mathrm{tr}\, S(y, x)S(x, y)$$

which cancels the second term in (6.20).

Collecting the above arguments, we find that we have justified the formal calculation in the introduction:

Theorem 6.5. For $j \neq k$

$$\partial^2 W/\partial q_j\, \partial q_k = -\int_{\Delta_j}\int_{\Delta_k} dx\, dy \operatorname{tr} S'(y, x) S'(x, y) \tag{6.21}$$

where $S' = [1 - S_0(m_f \chi_{D_R} + \phi_\delta \chi_\Lambda)]^{-1} S_0$.

But the results of Section 3 apply to S'. Appealing to Theorem 3.3, we conclude that:

Corollary 6.6. For $j \neq k$, $\partial^2 W/\partial q_j\, \partial q_k > 0$.

In the notation of this section, the definition (1.27) reads

$$\rho_{\delta,R} = \frac{1}{\int \rho(\tilde{K}, K)\, d\mu}\, \rho(\tilde{K}, K)$$

Thus, $\log \rho_{\delta,R} = W - \log \int \rho(\tilde{K}, K)\, d\mu$, and so (1.28) immediately follows.

7. CASE OF ZERO FERMI MASS

In this section we show that the results of the previous sections for $m_f > 0$ extend to the case $m_f = 0$. The basic Hilbert space is $\mathscr{H}_0 = L^2(|p| d^2 p) \otimes \mathbb{C}^2$; the Fermi two-point function is

$$S_0(x, y) = \frac{1}{(2\pi)^2} \int \frac{\not{p}}{p^2}\, e^{ip\cdot(x-y)}\, dp = \frac{1}{2\pi} \frac{\beta\cdot(x-y)}{|x-y|^2}$$

where the matrices β are given by (1.13); and

$$K' = S_0 \phi \chi_\Lambda \tag{7.1}$$

where we write K' instead of K since we intend to invoke Theorem 2.2.

It does not make sense to choose

$$:\operatorname{Tr} K'^\dagger K': = \frac{2}{(2\pi)^4} \int \frac{dp}{p^2} \int_\Lambda :\phi^2(x):\, dx$$

as the mass counterterm, since the infrared divergence of the infinite constant $\int (dp/p^2)$ does not cancel against anything. Accordingly we introduce

$$K = \chi_1 S_0 \phi \chi_\Lambda \tag{7.2}$$

where χ_1 is multiplication in momentum space by $\chi_1(p)$, the characteristic function of the set $\{p \mid |p| \geqslant 1\}$; and we choose as the counterterm

$$:\operatorname{Tr} K^\dagger K: = \frac{2}{(2\pi)^4} \int_{|p| \geq 1} \frac{dp}{p^2} \int_\Lambda :\phi^2(x):\, dx$$

In other words, we define $B(K', K)$ and the renormalized determinant $\rho(K', K)$ as in (2.30).

We first establish that $\rho(K', K) \in L^p(d\mu)$ by checking that K, K', and $V \equiv K' - K$ satisfy the hypotheses (a)–(d) of Theorem 2.2 for the (non-cutoff) boson field $\phi(x)$. The calculations closely follow those of the massive case except that we must exercise care over the small-momentum behavior (an occasional harmless logarithmic divergence occurs) and we have to fuss a little over momentum space cancellations since the counterterm is not "exact." Of course, in expressions involving K only there is no problem with small momentum.

(a) We must first show that $\|D^{-2\epsilon}K\|_{2,2}$, $\|D^{\epsilon}K\|_{4,4}$, and $\|B(K)\|_{L^2}$ are finite for some $\epsilon > 0$. We take $\epsilon < 1/4$. Now, as in the proof of Lemma 4.6,

$$\begin{aligned}\|D^{-2\epsilon}K\|_{2,2}^2 &= \int d\mu \operatorname{Tr} D^{-1}\phi\chi_\Lambda S_0^{\dagger}\chi_1 D^{1-4\epsilon}S_0\phi\chi_\Lambda \\ &= \text{const}\iint dp\, dq\, \frac{\chi_1(q)}{|p+q|\,|q|^{1+4\epsilon}}\int dk\, \frac{|\hat{\chi}_\Lambda(p+k)|^2}{\mu(k)^2}\end{aligned} \tag{7.3}$$

By Lemma C5,

$$\int dq\, \chi_1(q)|p+q|^{-1}|q|^{-1-4\epsilon} = O(|p|^{-4\epsilon})$$

and by Lemma C2,

$$\int dk\, |\hat{\chi}_\Lambda(p+k)|^2\mu(k)^{-2} = O\left[\prod_{i=0}^{1}(1+|p_i|)^{-1}\log(2+|p_i|)\right]$$

Hence

$$\|D^{-2\epsilon}K\|_{2,2}^2 \leqslant \text{const}\int dp\, |p|^{-4\epsilon}\prod_{i=0}^{1}(1+|p_i|)^{-1}\log(2+|p_i|) < \infty$$

Indeed, this bound is even more transparent in configuration space, and the bound $\|D^{\epsilon}K\|_{4,4} < \infty$ can be obtained easily in this way.

As for B,

$$B(K) = \text{const}\int_\Lambda\int_\Lambda b(x-y){:}\phi(x)\phi(y){:}$$

where [see (2.26)]

$$\begin{aligned}\hat{b}(k) &= \int\left(\frac{\chi_1(p)}{p^2} - \chi_1(p)\chi_1(p+k)\frac{(p+k)\cdot p}{(p+k)^2p^2}\right)dp \\ &= \int \chi_1(p)\left(\frac{1}{p^2} - \frac{(p+k)\cdot p}{(p+k)^2p^2}\right)dp \\ &\quad + \int \chi_1(p)[1-\chi_1(p+k)]\frac{(p+k)\cdot p}{(p+k)^2p^2}\,dp\end{aligned} \tag{7.4}$$

The first integral in (7.4) may be estimated exactly as in the massive case and is $O[\ln(2 + |k|)]$, whereas the second integral is $O[\mu(k)^{-1}]$. We conclude that

$$\hat{b}(k) = O[\ln(2 + |k|)] \tag{7.5}$$

and the argument continues as in the massive case.

As for hypothesis (iii) of Theorem 2.1,

$$\operatorname{Tr}(K + K^*)^2 = 2\operatorname{Tr}(K^2 + K^*K) = \text{const}\int_\Lambda \int_\Lambda dx\, dy\, w(x - y)\phi(x)\phi(y)$$

where [see (2.26)]

$$\hat{w}(k) = \int dp \left[\frac{1}{|p|\,|p + k|}\chi_1(p) - \frac{p\cdot(p + k)}{p^2(p + k)^2}\chi_1(p + k)\chi_1(p)\right]$$

We claim that $\hat{w}(k) = O(1)$ as in the massive case. To see this, we write, as in (7.4),

$$\hat{w}(k) = \int \chi_1(p)\left[\frac{1}{|p|\,|p + k|} - \frac{p\cdot(p + k)}{p^2(p + k)^2}\right] dp$$
$$+ \int \chi_1(p)[1 - \chi_1(p + k)]\frac{p\cdot(p + k)}{p^2(p + k)^2}\, dp$$

The second integral is $O[\mu(k)^{-1}]$, and the first, by scaling, is ($\hat{k} = k/|k|$)

$$\int \chi_1(|k|p)\left[\frac{1}{|p|\,|p + \hat{k}|} - \frac{p\cdot(p + k)}{p^2(p + \hat{k})^2}\right] dp$$
$$\leqslant \text{const} + \int_{|p|\geq 2}\left[\frac{1}{|p|\,|p + \hat{k}|} - \frac{p\cdot(p + \hat{k})}{p^2(p + \hat{k})^2}\right] dp \tag{7.6}$$

It is not hard to check that the integrand in (7.6) is $O(|p|^{-3})$ and so $\hat{w}(k) = O(1)$. Using the ultraviolet cutoff we introduced in Section 4, we obtain $\int d\mu \operatorname{Tr}(K_j + K_j^*)^2 \leqslant cj$ as in Lemma 4.10.

(b) We employ the same decomposition of K as in the massive case: $K = L + H$, where

$$L = \chi_2 S_0 \phi \chi_\Lambda \qquad \text{and} \qquad H = \theta_\sigma S_0 \phi \chi_\Lambda$$

with χ_2 the characteristic function of $\{k | 1 \leqslant |k| \leqslant \sigma\}$ and θ_σ that of $\{k|\ |k| \geqslant \sigma\}$. Then the kernel w_H of (2.27b) is given by [see (2.26)] |

$$\hat{w}_H(k) = \frac{2}{(2\pi)^3}\int\left[\chi_1(p)\frac{1}{p^2} - \theta_\sigma(p + k)|p + k|^{-1}|p|^{-1}\right] dp$$
$$= \frac{2}{(2\pi)^3}\int\left(\frac{\chi_1(p)}{p^2} - \frac{\chi_1(p)}{|p|}\frac{\theta_\sigma(p + k)}{|p + k|}\right) dp$$
$$+ \frac{2}{(2\pi)^3}\int\frac{\chi_1(p) - 1}{|p|}\frac{\theta_\sigma(p + k)}{|p + k|}\, dp$$

The second integral is $O(\sigma^{-1})$ and we estimate the first as in (2.29) by

$$\frac{\chi_1(p)}{|p|}\frac{\theta_\sigma(p+k)}{|p+k|} \leqslant \frac{1}{2}\left(\frac{\chi_1(p)}{p^2} + \frac{\theta_\sigma(p+k)}{(p+k)^2}\right)$$

to obtain

$$\hat{w}_H(k) \geqslant \frac{1}{(2\pi)^3}\int_{1\le p\le\sigma}\frac{dp}{p^2} + O(\sigma^{-1}) = \frac{1}{(2\pi)^2}\log\sigma + O(\sigma^{-1})$$

The usual bound $\hat{w}_H(k) = O[\ln(2+|k|)]$ leads to (2.31).

It remains to verify that L satisfies (2.27a). Now if $\zeta \in C_0^\infty$ satisfies $\zeta = 1$ on Λ, we can write

$$L = \chi_2 S_0 \zeta(p^2+1)\cdot T$$

where $T = (p^2+1)^{-1}\phi\chi_\Lambda$. By the unitary equivalence of $\mathscr{H}_0$ with $L^2 \otimes \mathbb{C}^2$, effected by multiplication in momentum space by $|p|^{1/2}$, we have

$$\begin{aligned}&\|\chi_2 S_0\zeta(p^2+1)\|^2_{\mathscr{C}_2(\mathscr{H}_0)}\\ &\quad= \text{const}\iint \big|\,|p|^{1/2}\chi_2(p)S_0(p)\hat{\zeta}(p-q)(q^2+1)|q|^{-1/2}\big|^2\,dp\,dq < \infty\end{aligned}$$

Likewise, by Lemma C3,

$$\|T\|^2_{2,2} = \text{const}\iiint |p|(p^2+1)^{-2}\frac{|\hat{\chi}_\Lambda(p-q-k)|^2}{\mu(k)^2}|q|^{-1}\,dp\,dq\,dk < \infty$$

so that by Hölder's inequality L satisfies (2.27a).

(c) Looking back at (7.3), we observe that $\|D^{-2\epsilon}K'\|_{2,2}$ is finite, i.e., there is no need for the cutoff χ_1. For our choice of ultraviolet cutoff we see that there is no significant change in the proof of Lemma 4.6, so we have the desired bound on $\|D^{-2\epsilon}\delta K_j'\|_{2,2}$. The bound on $\|D^\epsilon\delta K_j'\|_{4,4}$ is obtained by the same trick that was employed in Section 4.

From the definition (2.30),

$$\delta B_j = \text{const}\int_\Lambda\int_\Lambda b'(x-y){:}\phi(x)\delta\phi_j(y){:}$$

where, as in (7.4),

$$\begin{aligned}\hat{b}'(k) &= \int\left[\frac{\chi_1(p)}{p^2} - \frac{(p+k)\cdot p}{(p+k)^2p^2}\right]dp\\ &= \int \chi_1(p)\left[\frac{1}{p^2} - \frac{(p+k)\cdot p}{(p+k)^2p^2}\right]dp - \int_{p\le 1}\frac{(p+k)\cdot p}{(p+k)^2p^2}\,dp\\ &= O[\ln(2+|k|)] + O[\ln(2+|k|^{-1})]\end{aligned}$$

The additional logarithmic singularity at $k=0$ does not change the bound on $\|\delta B_j\|_{L^2}$ from the massive case.

(d) We calculate as in (7.3) that

$$\|D^{2\epsilon}V\|_{2,2}^2 = \text{const}\int\int dp\, dq\, \frac{1-\chi_1(q)}{|p+q|\,|q|^{1-4\epsilon}}\int dk\, \frac{|\hat{\chi}_\Lambda(p+k)|^2}{\mu(k)^2}$$

Since

$$\int_{|q|\le 1} dq\, |p+q|^{-1}|q|^{4\epsilon-1} = O[\mu(p)^{-1}]$$

it follows that $\|D^{2\epsilon}V\|_{2,2}^2 < \infty$. Similarly, if $\epsilon < \frac{1}{4}$, we find that $\|VD^{2\epsilon}\|_{2,2}^2 < \infty$.

In summary, we have verified the hypotheses of Theorem 2.2 for any choice of ϵ in $(0, \frac{1}{4})$. Hence:

Theorem 7.1. Suppose $m_f = 0$. Then $\rho(K', K)$, defined by (2.30), (7.1), and (7.2), is in $L^p(d\mu)$ for any $p < \infty$.

The lattice approximation follows at once since in the course of proving the above theorem we have established the same bounds on the "Fermi parts" of the momentum integrals that were used in Section 4. Consequently we have the FKG inequality (we are obviously spared the labors of Section 5):

Theorem 7.2. The FKG inequality of Theorem 6.8 extends to the case $m_f = 0$.

8. FKG INEQUALITY FOR PSEUDOSCALAR Y$_2$?

How does the analysis of the preceding sections for scalar Y$_2$ change in the case of pseudoscalar Y$_2$? The presence of the factor $\Gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in S [see (1.12)] does not affect any of the estimates. However, in the complex variables formulation, $\Gamma \approx i$, and so there are some changes: In the notation of Section 3 we find that relation (3.5) for $S' = (1 - S_0 h)^{-1}S_0$ is replaced by

$$S'(x, y) = -\begin{bmatrix} \text{Re } X_1(z, t) & \text{Re } X_2(z, t) \\ \text{Im } X_1(z, t) & \text{Im } X_2(z, t) \end{bmatrix} \tag{8.1}$$

where X_1 and X_2 are the fundamental generalized analytic functions for Eq. (3.8), i.e., solutions of (3.6a) with $c = 1$ and $c = -i$; but now

$$h(z) = -i\phi_\delta(z)\chi_\Lambda(z) - m_f\chi_{D_R}(z) \tag{8.2}$$

is no longer real as in (3.1b). As a result, (3.8) is no longer self-adjoint in the sense that (3.10′) and (3.10) agree. The adjoint fundamental solutions now satisfy

$$X_j'(z, t; \phi_\delta, m_f) = X_j(z, t; -\phi_\delta, m_f) \tag{8.3}$$

where we indicate the dependence on the "parameters" ϕ_δ and m_f. From (3.6a) we also have the obvious symmetry

$$X_1(z, t; \phi_\delta, m_f) = iX_2(z, t; -\phi_\delta, -m_f) \tag{8.4}$$

(3.11), (8.3), and (8.4) together yield, instead of (3.12),

$$\begin{aligned} X_1(z, t) &= \operatorname{Im} X_2^-(t, z) - i \operatorname{Im} X_1^-(t, z) \\ X_2(z, t) &= -\operatorname{Re} X_2^-(t, z) + i \operatorname{Re} X_1^-(t, z) \end{aligned} \tag{8.5}$$

where the superscript minus sign denotes a reversal in the sign of m_f, i.e., $X_j^-(t, z) = X_j(t, z; \phi_\delta, -m_f)$.

From (8.1) we calculate that

$$\begin{aligned} &\operatorname{tr} S'(x, y)S'(y, x) \\ &\quad = \operatorname{Re} X_1(z, t) \operatorname{Re} X_1(t, z) + \operatorname{Re} X_2(z, t) \operatorname{Im} X_1(t, z) \\ &\qquad + \operatorname{Im} X_1(z, t) \operatorname{Re} X_2(t, z) + \operatorname{Im} X_2(z, t) \operatorname{Im} X_2(t, z) \end{aligned}$$

Substituting from (8.5) for $X_j(z, t)$, we arrange that all the arguments are in the order (t, z):

$$\begin{aligned} &\operatorname{tr} S'(x, y)S'(y, x) \\ &\quad = \operatorname{Im} X_2^- \operatorname{Re} X_1 - \operatorname{Re} X_2^- \operatorname{Im} X_1 - \operatorname{Im} X_1^- \operatorname{Re} X_2 + \operatorname{Re} X_1^- \operatorname{Im} X_2 \\ &\quad = \operatorname{Im}(\bar{X}_1 X_2^- + \bar{X}_1^- X_2) \end{aligned}$$

As is painfully obvious, Lemma 3.8 applies only in the case $m_f = 0$ and we conclude that:

Lemma 8.1. Consider the lattice cutoff pseudoscalar Y_2 model with $m_f = 0$ and define S_0 by (1.12) with $m = 0$ and $\Gamma = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then $S' = (1 - S_0\phi_\delta\chi_\Lambda)^{-1}S_0$ is a bounded operator on $\mathscr{H}_0$ and satisfies

$$\operatorname{tr} S'(x, y)S'(y, x) < 0$$

There is an alternate way of deducing this result, which is worth mentioning. Consider the *scalar* two-point function defined by (3.1a) with $m = 0$:

$$S_s' = (1 - S_{0,s}\phi_\delta\chi_\Lambda)^{-1}S_{0,s} \tag{8.6}$$

where we attach the subscript s or p to denote scalar or pseudoscalar. Now let $\Gamma = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. It is easy to see that $e^{\theta\Gamma} = \cos\theta + \Gamma \sin\theta$. Take $\theta = \pi/4$. Then, since Γ anticommutes with β_j of (1.13), we find that

$$e^{-\theta\Gamma}\beta_j e^{\theta\Gamma} = \tfrac{1}{2}(1 - \Gamma)\beta_j(1 + \Gamma) = \beta_j\Gamma$$

Hence $S_{0,p} = e^{-\theta\Gamma}S_{0,s}e^{\theta\Gamma}$, and from (8.6) we have

$$S_p' = e^{-\theta\Gamma}S_s'e^{\theta\Gamma} \tag{8.7}$$

Lemma 8.1 is an obvious consequence of (8.7). It is not possible to obtain a relation like (8.7) when $m_f > 0$ and $S_p' = [1 - S_{0,p}(\phi_\delta\chi_\Lambda - m_f\Gamma)]^{-1}S_{0,p}$.

Upon removing the lattice cutoff as in Section 7, we complete the proof of Theorem 1.2 with the above lemma.

We do not know whether the basic inequality (1.19) holds for pseudoscalar Y_2 with $m_f > 0$. We can prove it for pseudoscalar Y_1; moreover, the few examples in higher dimensions for which we can compute explicitly have engendered in us a sense of cautious optimism. One such example is the case where $\phi\chi_\Lambda$ is a constant c on all of $\mathbb{R}^d$ (d arbitrary). In this case the β_j are $N \times N$ matrices, $N = 2^{[d/2]}$, satisfying

$$\beta_i\beta_j + \beta_j\beta_i = 2\delta_{i,j}$$

and Γ satisfies

$$\beta_i\Gamma + \Gamma\beta_i = 0, \qquad \Gamma^2 = -1 \tag{8.8b}$$

S is defined by (1.12) and

$$\begin{aligned} S' &= (1 - cS)^{-1}S = (-\beta\partial + m - c\Gamma)^{-1}\Gamma \\ &= (\beta\partial + m + c\Gamma)(-\Delta + m^2 + c^2)^{-1}\Gamma \end{aligned}$$

by (8.8). Hence

$$\begin{aligned} T &\equiv \operatorname{tr} S'(x, y)S'(y, x) \\ &= \frac{1}{(2\pi)^d}\int d^dp\, d^dq\, e^{i(p-q)\cdot z}\, \frac{\operatorname{tr}(\not p + m + c\Gamma)\Gamma(\not q + m + c\Gamma)\Gamma}{(p^2 + M^2)(q^2 + M^2)} \end{aligned}$$

where $z = x - y$ and $M^2 = m^2 + c^2$. It follows from (8.8) that

$$T = \frac{N}{(2\pi)^d}\int dp\, dq\, e^{i(p+q)\cdot z}\, \frac{c^2 - m^2 + p\cdot q}{(p^2 + M^2)(q^2 + M^2)} \tag{8.9}$$

Lemma 8.2. For all z, c, and m with $M \neq 0$, $z \neq 0$ we have $T < 0$.

Proof. Write $z = (z_0, \mathbf{z})$, where $\mathbf{z} \in \mathbb{R}^{d-1}$. By rotational symmetry we may suppose that $\mathbf{z} = 0$ and $z_0 > 0$. We evaluate the p_0 and q_0 integrals in (8.9) by residue calculus. The integrand has poles in the upper half-plane at $p_0 = i\mu(\mathbf{p}) \equiv i(\mathbf{p}^2 + M^2)^{1/2}$ and at $q_0 = i\mu(\mathbf{q})$. Hence

$$T = (2\pi)^{2-d}\int d\mathbf{p}\, d\mathbf{q}\, \exp\{-z_0[\mu(\mathbf{p}) + \mu(\mathbf{q})]\}\, \frac{c^2 - m^2 - \mu(\mathbf{p})\mu(\mathbf{q}) + \mathbf{p}\cdot\mathbf{q}}{4\mu(\mathbf{p})\mu(\mathbf{q})}$$

By the Cauchy–Schwarz inequality

$$\mu(\mathbf{p})\mu(\mathbf{q}) \geqslant M^2 + \mathbf{p}\cdot\mathbf{q} \geqslant c^2 - m^2 + \mathbf{p}\cdot\mathbf{q}$$

and so $T < 0$. ■

The analogous result holds for scalar Y_2 with d arbitrary. This is our evidence, referred to in Section 1, that the inequality (1.19) may hold in higher dimensions.

APPENDIX A. EQUIVALENCE OF FKG HYPOTHESES

The following "well-known"[18,19] lemma shows that the conditions (1.5) and (1.7) are essentially equivalent.

Lemma A1. Let $d\nu = e^w\, d^n q$ be a probability measure on $\mathbb{R}^n$ with $w \in C^2(\mathbb{R}^n)$. Then the condition

$$\partial^2 w/\partial q_j\, \partial q_k \geqslant 0, \qquad j \neq k \tag{A1}$$

holds if and only if $\rho = e^w$ satisfies the condition

$$\rho(p \vee q)\rho(p \wedge q) \geqslant \rho(p)\rho(q) \tag{A2}$$

where $(p \vee q)_i = \max(p_i, q_i)$ and $(p \wedge q)_i = \min(p_i, q_i)$.

Proof. Suppose that $p_i \leqslant q_i$ for $i = 1, 2,..., r$ and that $p_i \geqslant q_i$ for $i = r + 1,..., n$, where $1 \leqslant r \leqslant n - 1$. Then (A2) is equivalent to

$$\begin{aligned} &w(q_1,..., q_r, p_{r+1},..., p_n) - w(q)] \\ &\quad + [w(p_1,..., p_r, q_{r+1},..., q_n) - w(p)] \geqslant 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int_0^1 dt \sum_{j=r+1}^{n} (p_j - q_j) \\ &\qquad \times \frac{\partial w}{\partial q_j}(q_1,..., q_r, tp_{r+1} + (1 - t)q_{r+1},..., tp_n + (1 - t)q_n) \\ &\qquad - \int_0^1 dt \sum_{j=r+1}^{n} (p_j - q_j) \frac{\partial w}{\partial q_j}(p_1,..., p_r, tp_{r+1} + (1 - t)q_{r+1},...) \geqslant 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int_0^1 ds \int_0^1 dt \sum_{i=1}^{r} \sum_{j=r+1}^{n} (q_i - p_i)(p_j - q_j) \\ &\qquad \times \frac{\partial^2 w}{\partial q_i\, \partial q_j}(sq_1 + (1 - s)p_1,..., tp_n + (1 - t)p_n) \geqslant 0 \end{aligned}$$

Note that in the double sum $(q_i - p_i)(p_j - q_j) \geqslant 0$. We obviously obtain a similar conclusion whatever the ordering relations are for the p_i and q_i and so we obtain the stated equivalence of (A1) and (A2). ■

Thus our version of the FKG inequality, Theorem 1.1, is weaker than the version in Ref. 20 in that we assume that the density ρ is strictly positive and C^2. For applications the hypothesis (A1) is particularly convenient; but the requirement that ρ be strictly positive can be a nuisance and may necessitate procedures such as that of Section 5. It is fallacious to believe that the vanishing of ρ is a minor problem. In this connection we recommend an example of a non-strictly positive ρ given by Kemperman (Ref. 36, p. 329) in his exhaustive analysis of FKG and Holley–Preston[20] inequalities. In his example, ρ satisfies (1.7) pairwise, i.e., for any i and j, (1.7) holds provided $p_k = q_k$ for all $k \neq i, j$, but (1.6) fails [and hence so must (1.7) in general].

APPENDIX B. POINTWISE CONVERGENCE OF S_R'

Here we prove Theorem 5.1 concerning the pointwise convergence of $S_R'(x, y)$ to $S'(x, y)$. The two-point functions S_R and S_R' with spatially cutoff mass are defined in (5.2) and the non-cutoff functions S and S' are defined in (1.12) and (5.1).

We introduce the Banach algebra $\mathscr{B}$ of measurable, (2×2)-matrix-valued functions T on $\mathbb{R}^2 \times \mathbb{R}^2$ such that

$$\|T\| \equiv \operatorname*{ess\text{-}sup}_x \int dy\, |T(x, y)| < \infty$$

where $|T|$ denotes the HS norm on 2×2 matrices. Multiplication in $\mathscr{B}$ is defined by

$$(TU)(x, y) \equiv \int dx\, T(x, z)U(z, y)$$

We let $\mathscr{B}'$ be the Banach algebra with identity that is obtained by adjoining the δ-function to $\mathscr{B}$ in the canonical way.

Lemma B1. Let f be a bounded, measurable function on $\mathbb{R}^2$ with compact support. Then $S_R(x, y)f(y) \to S(x, y)f(y)$ in $\mathscr{B}$ as $R \to \infty$.

This lemma follows easily from the series calculation (5.14) for S_R. We omit the details. Note that Sf and $S_R f$ have integrable singularities at $x = y$ and hence are in $\mathscr{B}$.

Lemma B2. Suppose that $A_n \to A$ in $\mathscr{B}'$ and that $(1 - A)^{-1}$ exists in $\mathscr{B}'$. Then $(1 - A_n)^{-1}$ exists for sufficiently large n and converges to $(1 - A)^{-1}$ in $\mathscr{B}'$ as $n \to \infty$.

This elementary lemma holds in any abstract Banach algebra with identity as a consequence of the Neumann expansion

$$(1 - A_n)^{-1} = (1 - A)^{-1} \sum_{j=0}^{\infty} [(A_n - A)(1 - A)^{-1}]^j$$

Lemma B3. Let f be a bounded, measurable function with compact support. If $(1 - Sf)^{-1}$ is a bounded operator on $\mathscr{H} = \mathscr{H}_m$, then its kernel lies in $\mathscr{B}'$ and is the inverse of the kernel of $1 - Sf$ in $\mathscr{B}'$.

Proof. We write, as an operator identity,

$$(1 - Sf)^{-1} = \sum_{j=0}^{5} (Sf)^j + (1 - Sf)^{-1}(Sf)^6 = \sum_{j=0}^{5} (Sf)^j + (Sf)L(Sf) \tag{B1}$$

where $L = (1 - Sf)^{-1}(Sf)^4$. It is easy to verify that the first term in (B1) is in $\mathscr{B}'$. As for the second term, $L \in \mathscr{C}_1$ on $\mathscr{H}$ since $Sf \in \mathscr{C}_4$ on $\mathscr{H}$ (as can be seen by computing the appropriate trace in configuration space). Thus the standard decomposition, $L = \sum_n \alpha_n(f_n, \cdot)_{\mathscr{H}} g_n$, involves a summable sequence $\{\alpha_n\}$ with $\|f_n\|_{\mathscr{H}} = \|g_n\|_{\mathscr{H}} = 1$. Now

$$(SfLSf)(x, y) = \sum_n \alpha_n (Sfg_n)(x)\overline{f(y)}\overline{(S^\dagger Df_n)(y)}$$

so that

$$\|SfLSf\| \leqslant \sum_n |\alpha_n| \, \|Sfg_n\|_\infty \|f\|_2 \|S^\dagger Df_n\|_2 \tag{B2}$$

Clearly $\|S^\dagger Df_n\|_2 \leqslant \|f_n\|_2 \leqslant \text{const } \|f_n\|_{\mathscr{H}}$, and

$$\|Sfg_n\|_\infty \leqslant \|SfD^{-1/2}\|_{2,\infty} \|g_n\|_{\mathscr{H}}$$

where $\|\cdot\|_{2,\infty}$ denotes the operator norm from L^2 to L^∞. By the Schwarz inequality,

$$\|SfD^{-1/2}\|_{2,\infty}^2 \leqslant \sup_x \int \left| \int dz \, S(x, z) f(z) D^{-1/2}(z - y) \right|^2 dy \tag{B3}$$

We bound (B3) using Lemma C6 and the explicit small-distance behavior: $S(x, z) \sim |x - z|^{-1}$ and $D^{-1/2}(z - y) \sim |z - y|^{-3/2}$. Hence, by (B2), $(SfLSf)(x, y) \in \mathscr{B}$, and so, by (B1), $(1 - Sf)^{-1}(x, y) \in \mathscr{B}'$.

That $(1 - Sf)^{-1}(x, y)$ is the inverse of $(1 - Sf)(x, y)$ in $\mathscr{B}'$ follows from the obvious fact that the kernel of a product of bounded operators in $\mathscr{H}$ whose kernels lie in $\mathscr{B}'$ is given by the product in $\mathscr{B}'$ of the kernels. ■

Proof of Theorem 5.1. By (5.2c), $S_R' = (1 - S_R f)^{-1} S_R$, where $f \equiv \phi_\delta \chi_\Lambda$ is a bounded, measurable function with compact support. By Lemma B3, $(1 - Sf)^{-1}(x, y) \in \mathscr{B}'$, and by Lemmas B1 and B2, $(1 - S_R f)^{-1}(x, y) \to (1 - Sf)^{-1}(x, y)$ in $\mathscr{B}'$. Now let $h \in C_0^\infty(\mathbb{R}^2)$. Then by Lemma B1, $S_R(x, y)h(y) \to S(x, y)h(y)$ in $\mathscr{B}$. It follows that the kernel of $(1 - S_R f)^{-1} S_R h$ converges to the kernel of $(1 - Sf)^{-1} Sh$ in $\mathscr{B}$ as $R \to \infty$. Thus

$$\lim_{R \to \infty} \sup_x \int dy \, |S_R'(x, y) - S'(x, y)| \, |h(y)| = 0$$

and the theorem follows since h is arbitrary. ■

APPENDIX C. SOME STANDARD ESTIMATES

Here we collect and prove various estimates used in the main text. We shall denote all universal constants by the same letter c. The first estimates concern the function of $k \in \mathbb{R}$,

$$f(k) = \int_{-\infty}^{\infty} dp\,(1 + |p|)^{-\epsilon}(1 + |p + k|)^{-\alpha} \tag{C1}$$

Lemma C1. For $\alpha > 1$ and $0 \leqslant \epsilon \leqslant \alpha$,

$$f(k) \leqslant c(1 + |k|)^{-\epsilon}$$

Proof. We divide the integration in (C1) into the two regions, $|p| > |k|/2$ and $|p| \leqslant |k|/2$. In the first region we may extract the factor $(1 + |k|)^{-\epsilon}$ from the factor $(1 + |p|)^{-\epsilon}$; in the second region we use

$$(1 + |p + k|)^{-\alpha} \leqslant (1 + |k|/2)^{-\epsilon}(1 + |p|)^{-\alpha+\epsilon} \quad \blacksquare$$

Lemma C2. For $0 \leqslant \alpha, \epsilon \leqslant 1$ with $\alpha + \epsilon > 1$,

$$f(k) \leqslant \begin{cases} c(1 + |k|)^{1-\alpha-\epsilon} & \text{if} \quad \alpha, \epsilon < 1 \\ c(1 + |k|)^{1-\alpha-\epsilon}\log(2 + |k|) & \text{if} \quad \alpha = 1 \text{ or } \epsilon = 1 \end{cases}$$

Proof. $f(k)$ is obviously bounded uniformly on the interval $|k| \leqslant 1$. Now assume that $|k| > 1$ and set $\lambda = |k|$, $\hat{k} = \lambda^{-1}k$, $p = \lambda q$:

$$f(k) = \lambda^{1-\alpha-\epsilon}\int_{-\infty}^{\infty} dq\,(\lambda^{-1} + |q|)^{-\epsilon}(\lambda^{-1} + |q + \hat{k}|)^{-\alpha} \tag{C2}$$

If $\alpha, \epsilon < 1$, then $f \leqslant \lambda^{1-\alpha-\epsilon}\int dq\,|q|^{-\epsilon}|q + \hat{k}|^{-\alpha} = O(\lambda^{1-\alpha-\epsilon})$. If either $\alpha = 1$ or $\epsilon = 1$, it is easy to see that the integral in (C2) diverges logarithmically in λ as $\lambda \to \infty$. $\blacksquare$

Next we consider the function on $\mathbb{R}^2$

$$g(k, k') = \int_{-\infty}^{\infty} dp\,(1 + |p|)^{\epsilon}(1 + |p + k|)^{-\alpha}(1 + |p + k'|)^{-\beta} \tag{C3}$$

Lemma C3. For $0 \leqslant \alpha, \beta, \epsilon \leqslant 1$ with $\alpha + \beta - \epsilon > 1$,

$$g(k, k') \leqslant \begin{cases} c(1 + |k| + |k'|)^{\epsilon}(1 + |k' - k|)^{1-\alpha-\beta} & \text{if} \quad \alpha, \beta < 1 \\ c(1 + |k| + |k'|)^{\epsilon}(1 + |k' - k|)^{1-\alpha-\beta}\log(2 + |k' - k|) \\ \quad \text{if} \quad \alpha = 1 \text{ or } \beta = 1 \end{cases}$$

Proof. We make the change of variable $p \to p - k'$ in (C3) and assume without loss of generality that $|k'| \leqslant |k|$. Breaking up the integration into the regions $|p| \leqslant |k|$ and $|p| > |k|$, we obtain

$$\begin{aligned} g(k,k') &= \int dp\,(1+|p-k'|)^{\epsilon}(1+|p|)^{-\beta}(1+|p+k-k'|)^{-\alpha} \\ &\leqslant \int_{|p|\leq|k|} dp\,(1+|k|+|k'|)^{\epsilon}(1+|p|)^{-\beta}(1+|p+k-k'|)^{-\alpha} \\ &\quad + c\int_{|p|>|k|} dp\,(1+|p|)^{\epsilon-\beta}(1+|p+k-k'|)^{-\alpha} \end{aligned}$$

since when $|p| > |k|$ we have $|p - k'| \leqslant 2|p|$. Applying the preceding lemma to the remaining integrals, we obtain the desired conclusion. ■

In the course of proving Lemmas 4.4 and 4.9 we appealed to the following lemma with $b = 2\pi$:

Lemma C4. Take $b > 0$ and $0 \leqslant \alpha, \beta, \nu \leqslant 1$ with $\alpha + \beta + \nu > 2$. Then

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk'\,(1+|k|)^{-\alpha}(1+|k'|)^{-\beta}(1+|k'-k+b/\delta|)^{-\nu} = 0$$

Proof. We apply Lemma C2 to integrate out k' and bound the double integral by

$$c\int dk\,(1+|k|)^{-\alpha}(1-|k-b/\delta|)^{-\lambda} \tag{C4}$$

where $\lambda < \nu + \beta - 1$. By assumption we may choose $\lambda > 1 - \alpha \geqslant 0$. Once again by Lemma C2, (C4) is bounded by $(1 + |b|/\delta)^{-\epsilon}$, where $0 < \epsilon \leqslant \alpha + \lambda - 1$. ■

The next two lemmas state similar estimates for integrals on $\mathbb{R}^2$. The first follows by a scaling argument as in the proof of Lemma C2, and the second may be found on p. 39 of Ref. 25.

Lemma C5. If $0 \leqslant \epsilon < \alpha < 1$, then

$$\int d^2p\,(1+|p|)^{-1-\alpha}(1+|p+k|)^{-1+\epsilon} \leqslant c(1+|k|)^{\epsilon-\alpha}$$

Lemma C6. For $0 \leqslant \alpha, \beta < 2$ and $\Lambda \subset \mathbb{R}^2$ bounded

$$\int_\Lambda d^2z\,|z-x|^{-\alpha}|z-y|^{-\beta} \leqslant \begin{cases} c & \text{if } \alpha+\beta<2 \\ c\log(2+|x-y|^{-1}) & \text{if } \alpha+\beta=2 \\ c|x-y|^{2-\alpha-\beta} & \text{if } \alpha+\beta>2 \end{cases}$$

Our final estimate concerns the Sobolev spaces $\mathscr{H}_m$ defined in (1.10):

Lemma C7. Let $\zeta \in C_0^\infty(\mathbb{R}^2)$ and let $m > 0$. Then $f \mapsto \zeta f$ is a bounded mapping from $\mathscr{H}_0$ to $\mathscr{H}_m$.

Proof. Let $|\cdot|$ denote the usual Pythagorean norm on two-component vectors. We need to show that

$$\int dp\,(p^2 + m^2)|\widehat{\zeta f}(p)|^2 \leqslant c\int dp\,|p|\,|\hat{f}(p)|^2 \tag{C5}$$

Now

$$\begin{aligned}
&(p^2 + m^2)^{1/4}|\widehat{\zeta f}(p)| \\
&= \int (p^2 + m^2)^{1/4}|\hat{\zeta}(p-k)|\,|\hat{f}(k)|\,dk \\
&\leqslant \int [|p-k|^{1/2}|\hat{\zeta}(p-k)|\,|\hat{f}(k)| + |\hat{\zeta}(p-k)|\,|k|^{1/2}|\hat{f}(k)| \\
&\qquad + m^{1/2}|\hat{\zeta}(p-k)|\,|\hat{f}(k)|]\,dk
\end{aligned}$$

so that the left side of (C5) is bounded by

$$2\int dp\left[\int dk\,(|p-k|^{1/2} + m^{1/2})|\hat{\zeta}(p-k)|\,|\hat{f}(k)|\right]^2 + 2\int dp\left[\int dk\,|\hat{\zeta}(p-k)|\,|k|^{1/2}|\hat{f}(k)|\right]^2 \tag{C6}$$

By Young's inequality the second term in (C6) is smaller than $2\|\hat{\zeta}\|_1{}^2 \times \|\,|k|^{1/2}|\hat{f}(k)|\,\|_2{}^2$, as required. The first term in (C6) may be written as $\|h * \hat{f}\|_2{}^2$, where $h(p) = (|p|^{1/2} + m^{1/2})\hat{\zeta}(p)$. We let $f = f_1 + f_2$, where $\hat{f}_1 = \chi_1\hat{f}$, with $\chi_1(p)$ the characteristic function of $\{|p| \leqslant 1\}$. Then $|\hat{f}_2(p)| \leqslant |p|^{1/2}|\hat{f}(p)|$, so that

$$\begin{aligned}
\|h * \hat{f}\|_2 &\leqslant \|h * \hat{f}_1\|_2 + \|h * \hat{f}_2\|_2 \\
&\leqslant \|h\|_2\|\hat{f}_1\|_1 + \|h\|_1\|\hat{f}_2\|_2 \qquad \text{(Young's inequality)} \\
&\leqslant \|h\|_2\,\|\chi_1|p|^{-1/2}\|_2\,\|\,|p|^{1/2}|\hat{f}(p)|\,\|_2 + \|h\|_1\,\|\,|p|^{1/2}|\hat{f}(p)|\,\|_2 \\
&= c\|\,|p|^{1/2}|\hat{f}(p)|\,\|_2,
\end{aligned}$$

as required. ■

ACKNOWLEDGMENTS

We have benefited from useful conversations with Jürg Fröhlich, Ira Herbst, and Yong Moon Park.

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